

TREE INVARIANTS AND MILNOR LINKING NUMBERS WITH INDETERMINACY

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ABSTRACT. The paper concerns the *tree invariants* of string links, introduced by Kravchenko and Polyak and closely related to the classical Milnor linking numbers also known as $\bar{\mu}$ -invariants. We prove that, analogously as for $\bar{\mu}$ -invariants, certain residue classes of tree invariants yield link homotopy invariants of closed links. The proof is arrow diagrammatic and provides a more geometric insight into the indeterminacy through certain tree stacking operations. Further, we show that the indeterminacy of tree invariants is consistent with the original Milnor's indeterminacy. For practical purposes, we also provide a recursive procedure for computing arrow polynomials of tree invariants.

1. INTRODUCTION

Arrow polynomial formulas of Polyak and Viro [29] are a computationally attractive way to represent Vassiliev's finite type invariants of knots and links [34, 3, 1]. The input to such formula is a *Gauss diagram* G_L of a (based) link $L = L_1 \cup \dots \cup L_n$, $L_i : S^1 \mapsto \mathbb{R}^3$ in \mathbb{R}^3 , obtained from any plane projection of L , by drawing n disjoint oriented circles with basepoints, and marked (positive/negative) arrows connecting distinct points on the circles. Points of the i th circle correspond to values of a parameter for $L_i : S^1 \mapsto \mathbb{R}^3$, c.f. [10]. A positive/negative arrow between two points on the i th and j th circle is drawn, if and only if, for the corresponding parameter values the plane diagram of L has a positive/negative crossing, the arrow points from the underpass to the overpass. Alternatively, we may replace the components of G_L i.e. the circles, with vertical or horizontal oriented segments, we call *strings*, assuming that the beginning and end of each string is identified with the basepoint, Figure 1 illustrates this situation.

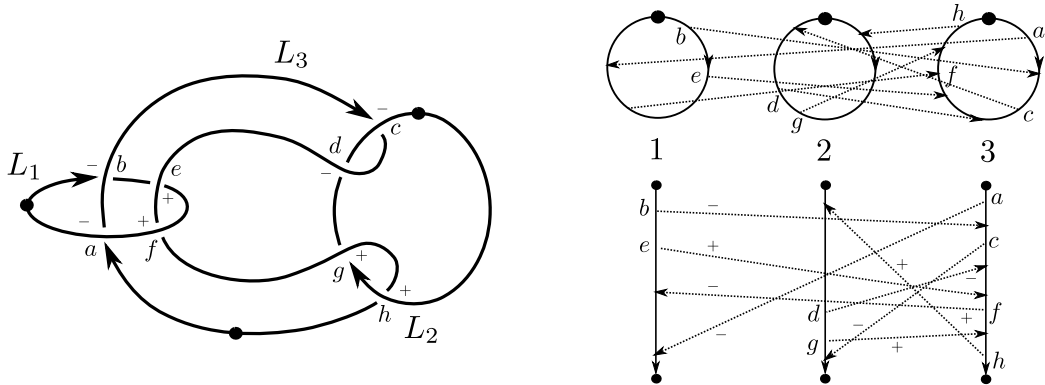


FIGURE 1. Gauss diagrams of Borromean rings (drawn with *circle* and respectively *string* components).

An *arrow diagram* A is an arbitrary unmarked diagram (i.e. arrows have no signs attached to them) analogous to a Gauss diagram but not necessarily obtained from a plane link diagram. The

arrow diagrams and Gauss diagrams can be paired as follows; for a Gauss diagram $G = G_L$ of the n -component link L , a *representation* of an arrow diagram A in G is an embedding of A into G mapping components of A to components of G , preserving the basepoints and arrow orientations. Define the *sign* of a representation $\phi : A \rightarrow G$ by¹

$$\text{sign}(\phi) = \prod_{\alpha \in A} \text{sign}(\phi(\alpha)), \quad (1.1)$$

where the $\text{sign}(\phi(\alpha))$ is a sign of the arrow $g = \phi(\alpha)$ in G . Then $\langle A, G \rangle$ stands for the sum

$$\langle A, G \rangle = \sum_{\phi: A \rightarrow G} \text{sign}(\phi), \quad (1.2)$$

taken over all embeddings $\phi : A \rightarrow G$ of A in G . An arrowhead of $\alpha \in A$ will be denoted by $h(\alpha)$ and the arrowtail by $t(\alpha)$. We write $\alpha \sim (i, j)$ or $g \sim (i, j)$ if α , resp. g , has its head on the i -component of A , and tail on the j -component. A formal sum of arrow diagrams $P = \sum_i c_i A_i$ with integer coefficients is known as an *arrow polynomial* [29], and $\langle P, G \rangle$ is defined from (1.1) and (1.2) by the linear extension. Theorem of Goussarov [10] shows that any finite type invariant v of knots can be expressed as $\langle P_v, \cdot \rangle$ for a suitable choice of the arrow polynomial P_v . Arrow polynomials of some low degree invariants have been computed in [27, 29, 30, 35]. For instance, the second coefficient of the Conway polynomial $c_2(K)$ of a knot K , represented by a Gauss diagram G_K is given by $\langle \bigotimes, G_K \rangle$ (c.f. [29]). Apart from low degree examples, the arrow diagram formulae are known for: the coefficients of the Conway [5] and the HOMFLY-PT polynomials, [6].

In the case of *string links* [11, 2], Kravchenko and Polyak [18] introduced a family of link homotopy invariants, called *tree invariants* which are closely related to the classical Milnor linking numbers, [25, 26, 21]. Recall that homotopy or link-homotopy is a weaker form of equivalence than isotopy

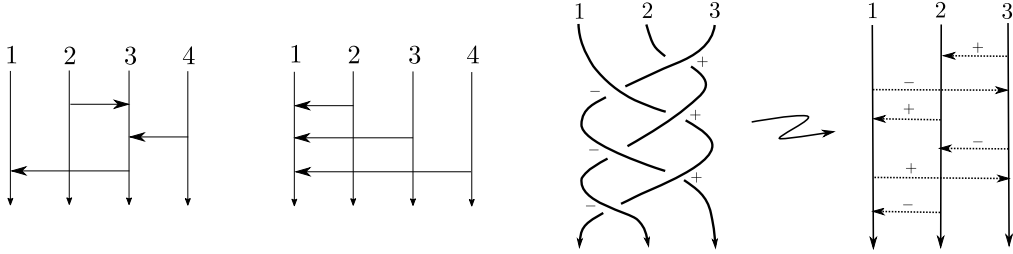


FIGURE 2. Sample tree diagrams in $\mathcal{A}(I; 1)$, $I = \{1, 2, 3, 4\}$ (left). Gauss diagrams of a Borromean string link (right).

which, in particular, allows self-crossing of strands. Specifically, Kravchenko and Polyak defined a family of (*planar*) *tree diagrams* on n components with a distinguished component j called a *trunk*, for $I = \{i_1, i_2, \dots, i_r\}$, $1 \leq i_1 < i_2 < \dots < i_r \leq n$ and $j \in I$, denote the set of these diagrams with leaves indexed by I and a trunk on j by² $\mathcal{A}(I; j)$, referring to Section 4 for further details. Paraphrasing the main theorem of [18] we have

Theorem 1.1 (Kravchenko and Polyak [18]). *Let $\ell = (\ell_1, \dots, \ell_n)$ be an n -component string link and G_ℓ its Gauss diagram. Consider the following arrow polynomial*

$$Z_{I; j} = \sum_{A \in \mathcal{A}(I; j)} \text{sign}(A) A, \quad (1.3)$$

¹arrows of A will be denoted by the greek letters: α, β, \dots , and arrows of G lowercase letters: g, h, \dots

²this is a slightly adjusted notation from [18] (Figure 2(left)), where the index of the trunk is also in I .

then

$$Z_{I;j}(\ell) = \langle Z_{I;j}, G_\ell \rangle = \sum_{A \in \mathcal{A}(I;j)} \text{sign}(A) \langle A, G_\ell \rangle, \quad (1.4)$$

defines a link-homotopy invariant of the string link ℓ , where $\text{sign}(A) = (-1)^q$ and q is given by number of arrows in A pointing to the right.

Invariants $Z_{I;j}(\ell)$ are called the *tree invariants* and are finite type string link invariants, [4]. As such, they must necessarily be determined by the classical μ -invariants of string links, [12]. In their paper [18] Kravchenko and Polyak show that for a particular order of leaves and the trunk, namely for $I = \{i_1, i_2, \dots, i_r\}$ and j such that $1 \leq j < i_1 < i_2 < \dots < i_r \leq n$, we have the identity

$$Z_{I;j}(\ell) = \mu_{I;j}(\ell). \quad (1.5)$$

The main theorem of the current paper is a generalization of Theorem 1.1 to closed n -component links. Analogous result by Polyak and Viro, Theorem 6 in [29] (see also [27]), concerns the case $n = 3$.

Main Theorem. *Suppose L is a based n -component link and G_L its Gauss diagram, for $I = \{i_1, \dots, i_r\}$, $1 \leq i_1 < \dots < i_r \leq n$ and $j = i_k$ the following quantity defines a homotopy invariant of L ;*

$$\overline{Z}_{I;j}(L) = \langle Z_{I;j}, G_L \rangle \mod \Delta_Z(I; j), \quad (1.6)$$

where

$$\Delta_Z(I; j) = \gcd\{\langle Z_{J;k}, G_L \rangle \mid J \subsetneq I; k \in J\}. \quad (1.7)$$

Further, the relation of $\overline{Z}_{I;j}$ -invariants and Milnor higher linking numbers i.e. $\overline{\mu}$ -invariants, [25, 26] is obtained in the following.

Corollary 1.2. *For $I = \{i_1, i_2, \dots, i_r\}$ and $j = i_1$ such that $1 \leq i_1 < i_2 < \dots < i_r \leq n$ we have*

$$\overline{Z}_{I;j}(L) = \overline{\mu}_{I-\{j\};j}(L). \quad (1.8)$$

Recall that the $\overline{\mu}$ -invariants are important invariants of homotopy links. In particular, they are capable of distinguishing 3-component homotopy links as proven in [25]. The indeterminacy of $\overline{\mu}$ -invariants was studied by several authors most notably by Levine, see [20, 21] and references therein. It is not known however, if an appropriate form of indeterminacy, would provide a complete set of numerical link homotopy invariants. More recently Kotori [16], defines $\overline{\mu}$ -invariants for Turaev's nanophrases [32], via Magnus expansion adapted to nanophrases. In contrast, our definition is a direct generalization of Kravchenko–Polyak tree invariants and geared towards further applications in geometric knot theory, [15]. For a related connection between trees and Milnor invariants of string links one may refer to the recent work in [17], and other relevant approaches, e.g. in [23] and [33].

The techniques presented in this paper are geometric and depend on certain natural tree diagram decompositions, related ideas can be found in the work of Östlund [27]. As a practical outcome we also obtain inductive formulas for tree invariants (previously computed directly for $n = 3$ and $n = 4$ in [18]) which yields a convenient computational algorithm.

The paper is organized as follows; In Section 3, we review the original construction of $\overline{\mu}$ -invariants for links and string links as presented in [26, 21]. Construction of tree invariants of string links is presented in Section 4, together with definitions of some natural tree stacking operations. Section

4 also includes the above mentioned algorithm for generating arrow polynomials of tree invariants for an arbitrary n . Proof of Main Theorem and Corollary 1.2 is provided in Section 6.

In the last paragraph of this introduction we mention that the applications of Milnor linking numbers are fairly broad; including distant areas such as topological fluid dynamics or plasma physics, c.f. [8, 19, 7, 14]. In the forthcoming paper: [15], we show how the arrow diagrammatic formulation of linking numbers can be applied to address a geometric question of Freedman and Krushkal [9], concerning estimates for thickness of n -component links.

2. ACKNOWLEDGEMENTS

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3. LINKING NUMBERS OF CLOSED LINKS AND STRING LINKS

In this section we review the construction of Milnor invariants, [25, 26]. Denote by $L = L_1 \cup \dots \cup L_n$ an n -component ordered, oriented link in S^3 . Recall that, any diagram of L yields the Wirtinger presentation of $\pi = \pi_1(S^3 - L)$, where the generators are *meridians*, one for each arc in the diagram and the relations are derived from the crossings in the diagram, [13]. Different meridians of a given component are conjugate to each other in π , a choice of basepoint on the j -component L_j indicates a preferred meridian m_j . In π we also distinguish the *parallels*, i.e. push-offs of L_j denoted by l_j satisfying $\text{lk}(l_j, L_j) = 0$. Consider $F = F(m_1, \dots, m_n)$, the free group generated by the preferred meridians $\{m_j\}$. In [25] Milnor proved that the universal homomorphism $F \mapsto \pi$ descends to an epimorphism of the lower central series quotients:

$$\phi : F/F_q \mapsto \pi/\pi_q, \quad \text{for any } q,$$

(in fact, due to the later result of Stallings [31], ϕ is an isomorphism). Recall that given a group G , the lower central series is given as $G_1 = G$, $G_2 = [G, G_1]$, \dots , $G_p = [G, G_{p-1}]$, \dots . Thus, for any q there exists $l_j^q \in F(m_1, \dots, m_n)$ representing the parallel l_j modulo the q th stage of the lower central series of π (i.e. $\phi(l_j^q F_q) = l_j \pi_q$). Every element of F can be regarded as a unit in³ $\mathbb{Z}\langle X_1, \dots, X_n \rangle$ via the *Magnus expansion*, which is a ring homomorphism

$$\begin{aligned} \theta_F : \mathbb{Z}F &\longrightarrow \mathbb{Z}\langle X_1, \dots, X_n \rangle, \\ m_i &\longmapsto 1 + X_i, \quad m_i^{-1} \longmapsto 1 - X_i + X_i^2 - X_i^3 + \dots \end{aligned} \tag{3.1}$$

embedding F into $\mathbb{Z}\langle X_1, \dots, X_n \rangle$ as its group of units. Given the j th parallel l_j , as above, we have the expansion

$$\theta_F(l_j^q) = 1 + \sum_{\{i_1, \dots, i_s\} \subset [n]; s \geq 1} \mu_{i_1 \dots i_s; j} X_{i_1} X_{i_2} \dots X_{i_s}, \quad [n] = \{1, \dots, n\}. \tag{3.2}$$

The coefficients $\mu_{\mathbf{I}; j} = \mu_{i_1, \dots, i_s; j}$ are defined for each ordered sequence of integers $\mathbf{I} = (i_1, \dots, i_s)$ $1 \leq i_k \leq n$. Following [26], let

$$\begin{aligned} \Gamma_\mu(\mathbf{I}; j) &= \{\mu(q_1, q_2, \dots, q_{r-1}; q_r) \mid \text{where } \{q_1, q_2, \dots, q_{r-1}, q_r\}, 2 \leq r < s \text{ ranges over} \\ &\quad \text{all subsequences of } (i_1, i_2, \dots, i_s, j) \text{ obtained by deleting at least one} \\ &\quad \text{of its elements and permuting the remaining elements cyclically}\}, \end{aligned} \tag{3.3}$$

³the ring of power series in n non-commuting variables X_i .

and

$$\bar{\mu}_{\mathbf{I};j} \equiv \mu_{\mathbf{I};j} \pmod{\Delta_\mu(\mathbf{I};j)}, \quad \Delta_\mu(\mathbf{I};j) = \gcd(\Gamma_\mu(\mathbf{I};j)). \quad (3.4)$$

In [26], Milnor proved that, for $s < q$, $\mathbf{I} = \{i_1, i_2, \dots, i_s\}$, the residue classes $\bar{\mu}_{\mathbf{I};j}$ are isotopy⁴ invariants of L , and if the indices in $\{\mathbf{I}, j\}$ are all distinct, $\bar{\mu}_{\mathbf{I};j}$ are link-homotopy invariants. The residue classes $\bar{\mu}_{\mathbf{I};j}(L) = \bar{\mu}_{\mathbf{I};j}$ are commonly known as the *Milnor linking numbers* or *$\bar{\mu}$ -invariants*, and $\Delta_\mu(\mathbf{I};j)$ is called the *indeterminacy*. One obvious property of $\bar{\mu}$ -invariants is equivariance under permutations $\gamma \in \Sigma_n$, i.e.

$$\bar{\mu}_{i_1 i_2 \dots i_r; j}(L^\gamma) = \bar{\mu}_{\gamma(i_1) \gamma(i_2) \dots \gamma(i_r); \gamma(j)}(L), \quad (3.5)$$

where L^γ is the link L with permuted components; $L_i^\gamma = L_{\gamma(i)}$. Further relations are proven in [26, p. 294], for example cyclic symmetry:

$$\bar{\mu}_{i_1, i_2, \dots, i_{r-1}; i_r}(L) = \bar{\mu}_{i_2, i_3, \dots, i_r; i_1}(L). \quad (3.6)$$

In fact, due to cyclic symmetry (3.6), we may consider a smaller set

$$\begin{aligned} \Gamma'_\mu(\mathbf{I};j) &= \{\mu(q_1, q_2, \dots, q_{r-1}; q_r) \mid \text{where } \{q_1, q_2, \dots, q_{r-1}, q_r\}, 2 \leq r < s \text{ ranges} \\ &\quad \text{over all proper subsequences of } (i_1, i_2, \dots, i_s, j)\}, \\ \text{and } \Delta'_\mu(\mathbf{I};j) &:= \gcd(\Gamma'_\mu(\mathbf{I};j)). \end{aligned} \quad (3.7)$$

From the basic properties of gcd, one obtains

$$\Delta'_\mu(\mathbf{I};j) = \Delta_\mu(\mathbf{I};j).$$

For *string links*, the construction of μ -invariants is completely analogous. Recall that an n -component string link (see Figure 2(right) for an example of a string link diagram) is a smooth embedding $\ell : \sqcup_{k=1}^n I_k \mapsto D^2 \times I$, of n copies I_1, \dots, I_n of the unit interval I into the cylinder $C = D^2 \times I$, such that each $\sigma_k = \sigma|_{I_k}$ is anchored at the bottom and top of the cylinder at fixed points $\{a_k\}$, i.e. for each $k = 1, \dots, n$;

$$\ell_k(0) = (a_k, 0), \quad \ell_k(1) = (a_k, 1).$$

Each string link ℓ can be closed up into a link $L = \widehat{\ell}$ by adding unlinked connecting strands outside the cylinder $D^2 \times I \subset \mathbb{R}^3$, this *closure* operation is denoted by $\widehat{}$ in [11]. As before, one considers the group $\pi = \pi_1(C \setminus \sigma)$, this time there is a canonical choice of *meridians* $\{m_k\}$ represented by loops in $D^2 \times \{1\}$ based at a fixed point $x_0 \in \partial(D^2 \times \{1\})$, with $\text{lk}(m_k, \ell_k) = +1$. One also defines *canonical parallels*; l_k as loops in C based at x_0 and closed up by fixed arcs in the boundary of C , with $\text{lk}(l_k, \ell_k) = 0$. Again each parallel l_j has its expansion (3.2), modulo the q -stage of the lower central series of F and π . Differently from the case of closed links, the coefficients $\mu_{\mathbf{I};j}(\ell) = \mu_{\mathbf{I};j}$ do not require the indeterminacy and yield the isotopy invariants of string links, [21] and for distinct indices in $\{\mathbf{I}; j\}$, they define link-homotopy invariants. Given a link $L = \widehat{\ell}$, obtained as the closure of a string link ℓ , we have the following identity, [21]:

$$\bar{\mu}_{\mathbf{I};j}(L) \equiv \mu_{\mathbf{I};j}(\ell) \pmod{\Delta_\mu(\mathbf{I};j)}. \quad (3.8)$$

The linking numbers $\mu_{\mathbf{I};j}$ significantly depend on the order of integers in $(\mathbf{I}; j) = (i_1 i_2 \dots i_r, j)$ (e.g., in general $\mu_{i_1 i_2 \dots i_r; j}(\ell) \neq \mu_{i_2 i_1 \dots i_r; j}(\ell)$.)

⁴in fact they are concordance invariants as follows from the result of Stallings in [31]

4. TREE INVARIANTS

In Section 1, we introduced planar tree diagrams and tree invariants, in this section we provide their formal definitions, following the source in [18].

Let⁵ $I = \{i_1, i_2, \dots, i_r\}$, $1 \leq i_1 < i_2 < \dots < i_r \leq n$ and $j \in I$, a *tree diagram* A with *leaves* on components numbered by I and a *trunk* on the j -component, is an arrow diagram which satisfies the following conditions:

- (d1) An arrowtail $t(\alpha)$ and an arrowhead $h(\alpha)$ of an arrow $\alpha \in A$ belong to different strings;
- (d2) There is exactly one arrow with an arrowtail on the i -th string, if $i \in I - \{j\}$ and no such arrow if $i \notin I - \{j\}$;
- (d3) All arrows have arrowheads on strings indexed by I ;
- (d4) All arrowheads proceed the (unique) arrowtail for each $i \in I - \{j\}$, as we follow the i -th string along its (downward) orientation.

Define the *degree* of an arrow diagram A to be the total number of arrows in the diagram, and note that it is always equal to $r - 1$. One may visualize any tree diagram A , as the rooted tree graph: $T(A)$ obtained from A by removing the part of each of the components that lies below the corresponding (unique) arrowtail (by (d2) and (d4)). Then, $T(A)$ is graph-isomorphic to a rooted tree with r leaves and the root on the j -component. Every tree is a planar graph, however we will refer to a tree diagram A as *planar*, if in its planar realization the order of leaves coincides with the initial ordering $i_1 < i_2 < \dots < i_l < j < i_{l+1} < \dots < i_r$ of the components. Note that the above axioms imply that every arrow $\alpha \in A$ is uniquely determined by its coordinates (i, j) . In Figure 3(right), there are two planar and a non-planar tree diagrams, together with the trees obtained from each of them. In the remaining part of this paper, for practical purposes, we will make no distinction between $T(A)$ and A . As in Section 1, $\mathcal{A}(I; j)$ stands for the set of all planar

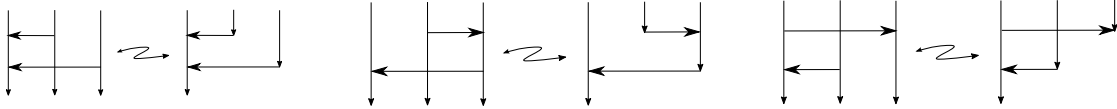


FIGURE 3. From a diagram A to a tree $T(A)$ (left). Planar and non-planar tree diagrams (middle and right).

tree diagrams with leaves on I and the trunk on the j -th component and $\mathcal{A}_j = \cup_I \mathcal{A}(I; j)$.

Remark 4.1 (Notation). Differently than in [18], we will treat the trunk of a tree diagram as one of its leaves. It makes only a small notational difference and simplifies further considerations. For instance $Z_{1,2,3,4;2} = Z_{134;2}$, where $Z_{134;2}$ agrees with Kravchenko and Polyak convention (which, whenever used, will skip the comma separators). Further, we also abbreviate

$$\mathcal{A}(n; j) := \mathcal{A}([n]; j), \quad Z_{n;j} := Z_{[n];j} = Z_{12 \dots \widehat{j} \dots n;j}.$$

Let G_ℓ be a Gauss diagram of a string link ℓ , see Figure 2(right) for an example. Kravchenko and Polyak [18], consider the following element of a quotient algebra $Dias(n)$ [18, p. 306] of $\mathbb{Z}\mathcal{A}_j$ where the product is defined via *tree grafting* and relations⁶ correspond to Loday's axioms of disassociative

⁵We assume a slightly different convention than in [18], see Remark 4.1.

⁶essentially encoding the Reidemeister moves

algebras [22],

$$Z_j(G_\ell) = \sum_{A \in A_j} \text{sign}(A) \langle A, G \rangle \cdot [A], \quad (4.1)$$

where $[A]$ denotes an equivalence class in $\text{Dias}(n)$, and

$$\text{sign}(A) = (-1)^q, \quad q = \# \text{ of arrows in } A \text{ pointing to the right}, \quad (4.2)$$

is called the *sign of the diagram* A .

Theorem 4.2 (Kravchenko and Polyak, [18]). *Let ℓ be an n -component string link and G_ℓ its Gauss diagram. Then $Z_j(\ell) = Z_j(G_\ell)$ is a $\text{Dias}(n)$ -valued homotopy invariant of ℓ .*

Since the equivalence class $[A]$ of a tree A , with a trunk on the j -th component, depends only on the set of its leaves, all $A \in \mathcal{A}(I, j)$ represent the same equivalence class and the sum given by

$$Z_{I,j}(G_\ell) = \sum_{A \in \mathcal{A}(I,j)} \text{sign}(A) \langle A, G_\ell \rangle$$

is a homotopy invariant as stated by Theorem 1.1 of Section 1. Invariants $\{Z_{I,j}(\ell)\}$ are finite type thus by [12, Corollary 6.4] they can be expressed as polynomials in Milnor μ -invariants of string links defined in Section 3. Because the index $I = (i_1, \dots, i_{n-1})$ for $\{\mu_{I;j}\}$ is ordered and $I = \{i_1, \dots, i_r\}$ in $Z_{I;j}$ has the increasing order, by assumption, in general: $Z_{I;j}(\ell) \neq \mu_{I-\{j\};j}(\ell)$. However, if $I = I$, i.e. I has the same order as I , then we have the following result

Theorem 4.3 (Kravchenko and Polyak, [18]). *Let ℓ be an ordered string link on n strings and let $1 \leq j = i_1 < i_2 < \dots < i_r \leq n$. Then*

$$Z_{I;j}(\ell) = \mu_{i_2 \dots i_r; j}(\ell). \quad (4.3)$$

(see also Corollary 4.5). The proof of Theorem 4.3 follows from the fact that the tree invariants $Z_{I,j}$ and $\mu_{I,j}$ satisfy the same skein relation (see [28]), and have the same normalization, i.e. $Z_{I;j}(\ell) = \mu_{I-\{j\};j}(\ell) = 0$ for all string links ℓ with the j -th component passing in front of all the others. Further relations for tree invariants were proven in [18, Proposition 4.2] and, for convenience, are stated below (where $I = \{i_1, \dots, i_r\}$, $1 \leq i_1 < \dots < i_r \leq n$).

(s1) For any $1 < k < r$ and $I^+ = \{i_1, \dots, i_{k-1}, i_k\}$, $I^- = \{i_k, i_{k+1}, \dots, i_r\}$,

$$Z_{I;i_k}(\ell) = Z_{I^+;i_k}(\ell) Z_{I^-;i_k}(\ell). \quad (4.4)$$

(s2) Let $\bar{\ell}$ be a string link obtained from ℓ by reflecting the ordering, i.e. $\bar{\ell}_i = \ell_{\bar{i}}$, where $\bar{i} = n + 1 - i$. Then,

$$Z_{I;j}(\bar{\ell}) = (-1)^r Z_{\bar{I};\bar{j}}(\ell). \quad (4.5)$$

(s3) Given a cyclic permutation $\sigma = (i_1 i_2 \dots i_r)$, let ℓ^σ be a string link with renumbered strings according to $\ell_i^\sigma = \ell_{\sigma(i)}$. Then

$$Z_{I;i_r}(\ell^\sigma) = Z_{I;i_1}(\ell). \quad (4.6)$$

Remark 4.4. Note that (s1) shows that a computation of $Z_{I;j}(\ell)$ with an arbitrary trunk $j \in I$, can be reduced to two cases: 1. the trunk is on the first component and 2. the trunk is on the last component.

At the end of this section let us exhibit some lower degree invariants $Z_{I;j}$ (for $n = 2$ and 3), as in [18, p. 308], and express them in terms of μ -invariants.

In the case where $n = 2$, we have the following tree invariants (using our notational conventions):

$$Z_{1,2;1}(\ell) = Z_{2;1}(\ell) = \left\langle \begin{array}{c} \leftarrow \\ \downarrow \\ \leftarrow \end{array}, G_\ell \right\rangle, \quad Z_{1,2;2}(\ell) = Z_{1;2}(\ell) = -\left\langle \begin{array}{c} \leftarrow \\ \downarrow \\ \rightarrow \end{array}, G_\ell \right\rangle, \quad (4.7)$$

which agree with the linking number:

$$Z_{2;1}(\ell) = \text{lk}(\ell_1, \ell_2) = -Z_{1;2}(\ell). \quad (4.8)$$

For diagrams with two arrows, we have

$$\begin{aligned} Z_{23;1}(\ell) &= \left\langle \begin{array}{c} \leftarrow \quad \leftarrow \\ \downarrow \quad \downarrow \\ \leftarrow \quad \leftarrow \end{array} - \begin{array}{c} \leftarrow \quad \rightarrow \\ \downarrow \quad \downarrow \\ \leftarrow \quad \rightarrow \end{array} + \begin{array}{c} \leftarrow \quad \leftarrow \\ \downarrow \quad \downarrow \\ \leftarrow \quad \leftarrow \end{array}, G_\ell \right\rangle, \quad Z_{13;2}(\ell) = -\left\langle \begin{array}{c} \leftarrow \quad \leftarrow \\ \downarrow \quad \downarrow \\ \leftarrow \quad \leftarrow \end{array} + \begin{array}{c} \rightarrow \quad \rightarrow \\ \downarrow \quad \downarrow \\ \rightarrow \quad \rightarrow \end{array}, G_\ell \right\rangle, \\ Z_{12;3}(\ell) &= \left\langle \begin{array}{c} \leftarrow \quad \rightarrow \\ \downarrow \quad \downarrow \\ \leftarrow \quad \rightarrow \end{array} - \begin{array}{c} \leftarrow \quad \leftarrow \\ \downarrow \quad \downarrow \\ \leftarrow \quad \leftarrow \end{array} + \begin{array}{c} \rightarrow \quad \rightarrow \\ \downarrow \quad \downarrow \\ \rightarrow \quad \rightarrow \end{array}, G_\ell \right\rangle. \end{aligned} \quad (4.9)$$

As shown in [18, Proposition 4.2], we have the following identities for the above invariants

$$\begin{aligned} Z_{23;1}(\ell) &= \mu_{23;1}(\ell), \\ Z_{13;2}(\ell) &= Z_{1;2}(\ell)Z_{3;2}(\ell) = -\mu_{1;2}(\ell)\mu_{3;2}(\ell), \\ Z_{12;3}(\ell) &= Z_{12;3}((\ell^{\sigma^{-1}})^\sigma) = Z_{23;1}(\ell^{\sigma^{-1}}) = \mu_{23;1}(\ell^{\sigma^{-1}}) = \mu_{\sigma(2)\sigma(3);\sigma(1)}(\ell) = \mu_{31;2}(\ell) \end{aligned} \quad (4.10)$$

The first equation is just (4.3). In the second identity, we applied **(s2)** and (4.8), in the third identity **(s3)** and (3.5) with $\sigma = (1\ 2\ 3)$. The invariants $\mu_{31;2}(\ell)$ and $\mu_{23;1}(\ell)$ are not equal and in general differ by a sign and a sum of products of pairwise linking numbers of ℓ . The computation in (4.10) can be easily generalized as follows.

Corollary 4.5. *Let $I = \{i_1, \dots, i_r\}$, $1 \leq i_1 < \dots < i_r \leq n$, if $1 < k < r$ then*

$$Z_{I; i_k}(\ell) = \mu_{i_3 i_4 \dots i_k i_1; i_2}(\ell) \mu_{i_{k+1} \dots i_r; i_k}(\ell). \quad (4.11)$$

In particular for $k = 1$ we obtain $Z_{i_2 \dots i_r; i_1}(\ell) = \mu_{i_2 \dots i_r; i_1}(\ell)$ and for $k = r$: $Z_{i_1 \dots i_{r-1}; i_r}(\ell) = \mu_{i_3 i_4 \dots i_k i_1; i_2}(\ell)$.

We end this section with a computational example.

Example 4.6. Let ℓ be the string link shown in Figure 2(right) along with its Gauss diagram G_ℓ , and let us compute $Z_{1,2,3;1}(\ell) = Z_{23;1}(\ell)$ using identities in (4.9). Note that G_ℓ contains three subdiagrams of the first type, two of which contribute +1 and one of which contributes -1. Also note that G_ℓ does not contain any subdiagrams of the other two types, as a result $Z_{23;1}(\ell) = 1$.

5. INDUCTIVE CONSTRUCTION OF TREE INVARIANTS

We begin by introducing a certain useful operation, we refer to as *tree stacking*, this operation is related but different from grafting which defines the product of the associative algebra $\text{Dias}(n)$, introduced earlier. Apart from the fact, that tree stacking is our primary tool in proving Main Theorem, it is essential to obtain an inductive description of planar trees, which is presented in the second part of this section.

Recall, that $\mathcal{A}(n; r)$ is the set of planar tree diagrams satisfying axioms **(d1)**–**(d4)** of Section 4. Tree stacking operations are indexed by the leaves and defined as

$$\begin{aligned} \prec_k: \mathcal{A}(n; r) \times \mathcal{A}(m; s) &\longrightarrow \mathcal{A}(m + n - 1; t), \\ (P, Q) &\longrightarrow P \prec_k Q, \end{aligned} \quad (5.1)$$

where $k \in [n]$ indexes a fixed leaf of a tree in $\mathcal{A}(n; r)$ and $P \prec_k Q$ is simply obtained by “gluing” the trunk of Q on the k th leaf of P , as shown in Figure 4. The index t of the trunk of $P \prec_k Q$ is determined as follows:

$$t = \begin{cases} r; & \text{if } r < k, \\ r + m - 1; & \text{if } r > k, \\ r + s - 1; & \text{if } r = k. \end{cases} \quad (5.2)$$

Operations \prec_k are clearly well defined, i.e. produce planar trees on $n + m - 1$ leaves with the

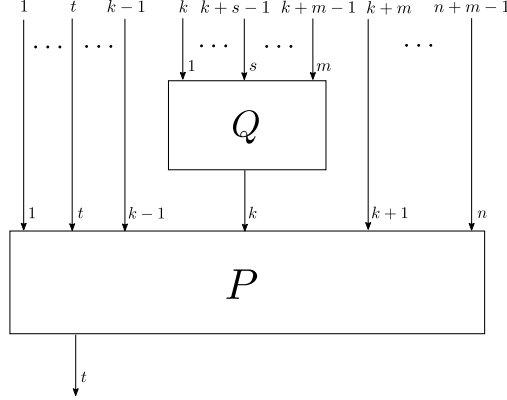


FIGURE 4. Stacking $Q \in \mathcal{A}(m; s)$ onto the k^{th} leaf of $P \in \mathcal{A}(n; r)$ yields $P \prec_k Q \in \mathcal{A}(m + n - 1; t)$.

trunk on t . We also set

$$P \prec_k \emptyset = P, \quad \emptyset \prec_k P = P.$$

Given $P \in \mathcal{A}(I; r)$, define

$$\begin{aligned} r(P) &= \text{number of leaves of } P \text{ to the right of the trunk,} \\ l(P) &= \text{number of leaves of } P \text{ to the left of the trunk.} \end{aligned} \quad (5.3)$$

And the *total number of leaves*:

$$|P| = r(P) + l(P) + 1.$$

It will be useful, to keep track of leaves indexing in P and Q after stacking. For $Z = P \prec_k Q$, $n = |P|$, and $m = |Q|$, we define the following multindices (as on Figure 4)

$$\begin{aligned} I(P; Z) &= [1, \dots, k-1] \cup \{k+s-1\} \cup [k+m, \dots, m+n], \\ I(Q; Z) &= [k, \dots, k+m-1], \end{aligned} \quad (5.4)$$

where $I(X; Z)$ indexes the leaves of X in Z . Suppose a tree diagram $A \in \mathcal{A}(n; t)$ admits a decomposition (where the parenthesis are inserted in an arbitrary way)

$$A = B_1 \prec_{b_1} B_2 \prec_{b_2} \dots \prec_{b_{k-1}} B_k, \quad B_i \in \mathcal{A}(m_i; t_i). \quad (5.5)$$

Let $j \in [n]$ the index of j th leaf component of A (j -component in short), we say that the j th leaf *stems* from B_i 'th factor in A , if and only if, it was added at the moment of stacking the B_i 'th factor in (5.5). Denote

$$\begin{aligned} i(j, A; B_i) &= \text{the index of } j\text{th leaf of } A \text{ in the } [m_j] \text{ index interval of } B_j, \\ i(k, B_i; A) &= \text{the index of } k\text{th leaf of } B_j \text{ in the index interval } [n] \text{ of } A. \end{aligned} \quad (5.6)$$

Note that $i(j, A; B_i)$ is only well defined for $j \in I(B_i; A)$, i.e. j th leaf stems from B_i . From (4.2) we also obtain the following sign identity,

$$\text{sign}(P \prec_k Q) = \text{sign}(P) \text{sign}(Q). \quad (5.7)$$

Building blocks of trees in $\mathcal{A}(n; r)$ are the unique elements of $\mathcal{A}(2; 1)$ and $\mathcal{A}(2; 2)$ shown on Figure 5 and called *elementary trees*. Whenever it is clear from the context the horizontal arrow of e or

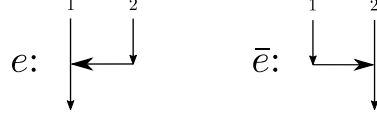


FIGURE 5. Elementary trees $e \in \mathcal{A}(2; 1)$ and $\bar{e} \in \mathcal{A}(2; 2)$.

\bar{e} will also be denoted by e or \bar{e} . The following lemma conveys a basic fact that trees can be recursively constructed by adding leaves.

Lemma 5.1. *Suppose $A \in \mathcal{A}(n; r)$, then there is $k \in [n - 1]$ such that*

$$A = A' \prec_k e, \quad \text{for } A' \in \begin{cases} \mathcal{A}(n - 1; r), & k \geq r; \\ \mathcal{A}(n - 1; r - 1), & k < r; \end{cases} \quad (5.8)$$

or

$$A = A' \prec_k \bar{e}, \quad \text{for } A' \in \begin{cases} \mathcal{A}(n - 1; r), & k > r; \\ \mathcal{A}(n - 1; r - 1), & k \leq r. \end{cases} \quad (5.9)$$

Proof. We will induct with respect to $n \geq 2$. For $n = 2$, the claim is obvious, since $\mathcal{A}(1; 1) = \emptyset$, $(n - 1) < 2$ and the only tree in $\mathcal{A}(2; 1)$ is given by $\emptyset \prec_1 e = e$ and in $\mathcal{A}(1; 2)$ the only tree is $\emptyset \prec_1 \bar{e} = \bar{e}$. For the inductive step, note that for any tree diagram $A \in \mathcal{A}(n + 1; r)$, in its planar realization as shown on Figure 3, horizontal arrows can be pushed up or down, so that there is at most one arrow at each vertical level of the diagram. Ordering the arrows, from the top to bottom, we let $\alpha_{top} \sim (i, j)$ be a top arrow. The top arrow has the obvious property that there are no arrowheads above $h(\alpha_{top})$ along the i -component, and equally are no arrowheads above $t(\alpha_{top})$ along the j -component of A (by (d2) there can be no arrowtail above an arrowhead). Note that α_{top} generally depends on the choices of heights, i.e. the vertical ordering of horizontal arrows along components. Because of planarity, we claim that α_{top} is “short” i.e.

$$j = \begin{cases} i + 1, & \text{if } i < j; \\ i - 1, & \text{if } i > j. \end{cases} \quad (5.10)$$

Indeed, if $i < j$ and $j > i + 1$, then there is a k -component, such that $i < k < j$. The vertical edge of k -component, would have to intersect the edge α_{top} , because the first arrowhead along this component is below $h(\alpha_{top})$. This contradicts planarity of A and proves the first case of (5.10), the second case of (5.10) can be shown analogously.

Now, thanks to the property (d4), there is no arrowhead/tail along the j -component, below the tail of α_{top} . Therefore, removing the vertical edge corresponding to that component together with α_{top} , yields a tree A' which is a subgraph of A . From the definition of \prec_k in (5.1) we obtain (5.8) or (5.9) proving the inductive step. \square

Clearly, the expansions: $A' \prec_k e$, and $A'' \prec_j \bar{e}$, for some A' and A'' may produce isomorphic diagrams. Lemma 5.1 gives an inductive definition of sets $\mathcal{A}(n; r)$ as follows

$$\begin{aligned} \mathcal{A}(1; 1) = \emptyset, \quad \mathcal{A}(n; r) = & \{A \prec_k e, A \prec_j \bar{e} \mid A \in \mathcal{A}(n-1; r-1), k=1, \dots, r-2; j=1, \dots, r-1\} \\ & \cup \{A \prec_k e, A \prec_j \bar{e} \mid A \in \mathcal{A}(n-1; r), k=r, \dots, n-1; j=r+1, \dots, n-1\}. \end{aligned}$$

The above formula provides an algorithm for computing $\mathcal{A}(n; r)$ recursively, and thus obtaining polynomials $Z_{n;r}$ given in (1.4). Figures 6 through 9 demonstrate the algorithm, by computing $\mathcal{A}(3; 1)$ and $\mathcal{A}(4; 1)$ and yield the arrow polynomial $Z_{1,2,3,4;1} = Z_{234;1}$. Figure 6 shows how the set $\mathcal{A}(3; 1) = \{A, B, C\}$ is obtained from expansions of the diagram $e \in \mathcal{A}(2; 1)$, where $A = e \prec_2 e$, $B = e \prec_1 e$, $C = e \prec_2 \bar{e}$. Figure 6 shows the corresponding arrow polynomial $Z_{23;1} = A + B - C$, see Equation (4.9).

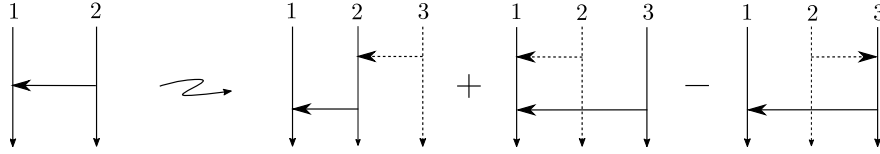


FIGURE 6. Obtaining $\mathcal{A}(3; 1) = \{A, B, C\}$ from $\mathcal{A}(2; 1) = \{e\}$.

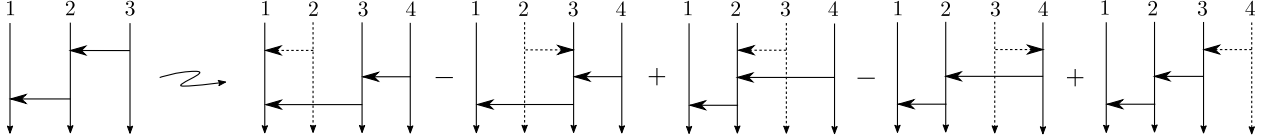


FIGURE 7. Applying elementary expansions $\prec e$ and $\prec \bar{e}$ to $A \in \mathcal{A}(3; 1)$.

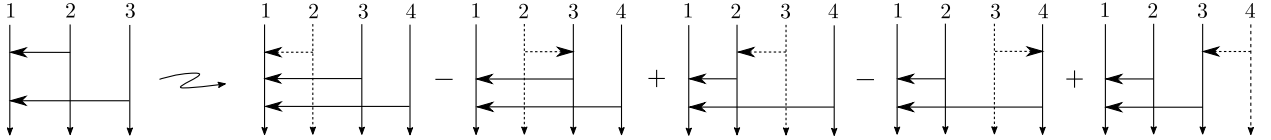


FIGURE 8. Applying elementary expansions $\prec e$ and $\prec \bar{e}$ to $B \in \mathcal{A}(3; 1)$.

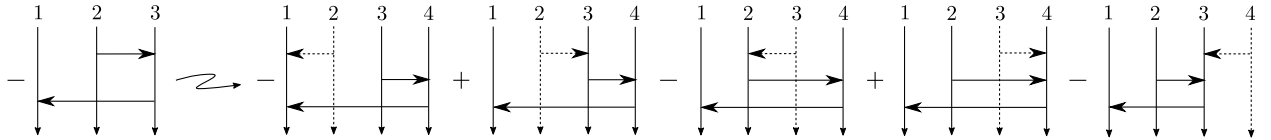


FIGURE 9. Applying elementary expansions $\prec e$ and $\prec \bar{e}$ to $C \in \mathcal{A}(3; 1)$.

Expanding the elements of $\mathcal{A}(3; 1)$ and eliminating duplicates yields $\mathcal{A}(4; 1)$. Observe that the second to last term in Figure 8 expansion is isomorphic to the first term of expansion in Figure 9, and the last term in Figure 8 expansion is isomorphic to the first term in Figure 7. Removing these duplicates yields the following arrow polynomial formula for the $Z_{234;1}$ invariant, which agrees with

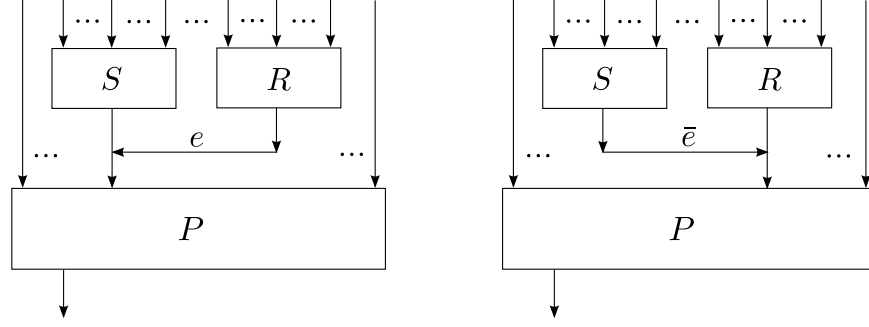


FIGURE 12. Decomposition of $A \in \mathcal{A}(n; t)$ with respect to the arrow α corresponding to e (left) or \bar{e} (right).

where $v = i(i, A; P)$ and e (or \bar{e} respectively) corresponds to α in A .

Proof. The proofs of (5.13) and (5.14) are analogous and follow by induction with respect to n , thus we provide details only in the case (5.13). The base case $n = 2$ follows immediately by choosing $P = R = S = \emptyset$. For the inductive step, suppose (5.13) is valid for n , and pick any $A \in \mathcal{A}(n+1; t)$. By Lemma 5.1, one of the following holds for some $k \in [n-1]$:

$$\begin{aligned} 1^\circ. & A = A' \prec_k e, \\ 2^\circ. & A = A' \prec_k \bar{e}, \quad \text{for } k > 1. \end{aligned} \tag{5.15}$$

In the case 1° ; if $\alpha = e$, $\alpha \sim (i, j)$ then $i = k$, $j = k+1$ and (5.13) is obtained by setting $P = A'$, $Q = e$, $R = \emptyset$, $S = \emptyset$. If α is different from e in (5.15), there must be $\alpha' \in A'$, which becomes α in A , after stacking either e or \bar{e} , according to (5.15). Let (i', j') be coordinates of α' , then i' equals either $i-1$ or i . By the inductive hypothesis applied to A' with α' we obtain

$$A' = P' \prec_{v'} Q', \quad Q' = (e \prec_2 R') \prec_1 S', \tag{5.16}$$

for certain P' , Q' , R' , S' and $v' = i(i', A'; P')$ where e in (5.16) corresponds to α' . Using Lemma 5.2 and (5.16), we have the following obvious subcases; for $x = e$ or $x = \bar{e}$ as in (5.15):

(a) for $k \in I(P'; A')$ and $k \notin I(Q'; A')$ we obtain (5.13) by defining

$$P = P' \prec_w x, \quad Q = Q', \quad w = i(k, A'; P').$$

(b) for $k \in I(S'; A')$ we obtain (5.13) by defining $P = P'$,

$$R = R', \quad S = S' \prec_w x, \quad w = i(k, A'; S').$$

(c) for $k \in I(R'; A')$ we obtain (5.13) by defining

$$P = P', \quad S = S', \quad R = R' \prec_w x, \quad w = i(k, A'; R'). \quad \square$$

6. PROOF OF MAIN THEOREM

Recall the statement of Main Theorem; Given a based n -component link L in \mathbb{R}^3 and its Gauss diagram G_L , the following quantity defines a homotopy invariant of L :

$$\bar{Z}_{I;j}(L) = \langle Z_{I;j}, G_L \rangle \mod \Delta_Z(I; j), \tag{6.1}$$

for $I = \{i_1, i_2, \dots, i_r\}$, $1 \leq i_1 < i_2 < \dots < i_r \leq n$ and $j = i_k$, $1 \leq k \leq r$, where

$$\Delta_Z(I; j) = \gcd\{\langle Z_{J;k}, G_L \rangle \mid J \subsetneq I, k \in J\}. \tag{6.2}$$

Apart from the presence of the indeterminacy $\Delta_Z(I; j)$ in (6.1) the main point of the above formula is that the tree polynomial $Z_{I;j}$ is evaluated on a Gauss diagram of a link G_L rather than a string link G_ℓ as in Theorem 1.1.

The proof of Main Theorem is divided into two parts; In the first part we analyze the difference $\langle Z_{I;j}, G_L \rangle - \langle Z_{I;j}, G'_L \rangle$, where G'_L is a Gauss diagram of L obtained after moving the basepoint of a component past an over/undercrossing. A similar basepoint change argument can be found in the work of Östlund, [27], for $n = 3$, but rather than using Östlund's *diagram fragments* we perform direct computations with tree diagrams and their suitable tree decompositions as stated in Lemma 5.3. In the second part of the proof, we show invariance under the Reidemeister moves, where the argument is analogous as the one by Kravchenko and Polyak in Theorem 4.2 of [18] and is included mainly for completeness of the exposition.

6.1. Basepoint change. Let us review our basic notation first; $G = G_L$ be a Gauss diagram of the n -component based link L . For convenience we will always visualize G as a string rather than a circular Gauss diagram; see Figure 1. A *representation* of a tree diagram $A \in \mathcal{A}(m; k)$ in G is an embedding $\phi : A \rightarrow G$ of A into G mapping components of A to the strings of G , preserving the endpoints and arrow orientations. Recall that arrows of A are denoted by the greek letters α, β, \dots , and arrows of G are denoted by lowercase letters: g, h, \dots . Further, the pairing $\langle A, G \rangle$ is given by the sum

$$\langle A, G \rangle = \sum_{\phi: A \rightarrow G} \text{sign}(\phi), \quad \text{where} \quad \text{sign}(\phi) = \prod_{\alpha \in A} \text{sign}(\phi(\alpha)), \quad (6.3)$$

taken over all representations of A in G . The following notation for a partial sum will be used frequently

$$\langle A, G \rangle_{\text{cond.}} = \sum_{\substack{\phi: A \rightarrow G; \\ \phi \text{ satisfies } \text{cond.}}} \text{sign}(\phi), \quad (6.4)$$

where the sum is only over those representations which satisfy a given condition: **cond.**. For instance given an arrow $g \in G$ and an arrow $\alpha \in A$, we write

$$\langle A, G \rangle_{\alpha \mapsto g} = \sum_{\substack{\phi: A \rightarrow G; \\ \phi(\alpha) = g}} \text{sign}(\phi), \quad \langle A, G \rangle_{\alpha \not\mapsto g} = \sum_{\substack{\phi: A \rightarrow G; \\ \phi(\alpha) \neq g}} \text{sign}(\phi). \quad (6.5)$$

Note that, for any α and g ,

$$\langle A, G \rangle = \langle A, G \rangle_{\alpha \mapsto g} + \langle A, G \rangle_{\alpha \not\mapsto g}. \quad (6.6)$$

If a Gauss diagram G has n -components, $A \in \mathcal{A}(m; q)$, $m \leq n$, and $I = \{i_1, \dots, i_m\}$, $1 \leq i_1 < i_2 < \dots < i_m \leq n$, we denote by $G(I)$ the subdiagram of G obtained by removing components (together with the adjacent arrows) of G which are not in I , and define

$$\langle A, G(I) \rangle = \sum_{\phi: A \rightarrow G; \phi(A) \subset G(I)} \text{sign}(\phi). \quad (6.7)$$

From the definitions presented in Section 1, $\langle A, G(I) \rangle$, equals $\langle A', G \rangle$ where $A' \in \mathcal{A}(I; i_q)$ is obtained from A by simply indexing the leaves and the trunk by $(I; i_q)$. Each step in our analysis is pictured on Figures 13 through 19, where G (left) and G' (right) differ by a basepoint move, i.e. the arrow $g \sim (i, j)$ in G becomes g' in G' and the “dashed” part of the diagrams is common for both G and G' . We consider four cases $(\iota) - (\nu)$ as the basepoint of the i th or j th string passes through the

arrowhead/tail of g . The basepoint passes are denoted by (a) and (b) in each of the Figures 13–19. The following result characterizes the difference $\langle Z_{n;t}, G \rangle - \langle Z_{n;t}, G' \rangle$, for $1 \leq t \leq n$.

Lemma 6.1. *Let G be a Gauss diagram of a based closed link L , and consider a Gauss diagram G' obtained from G via a moving a basepoint past a crossing in each case: (ι) – (ιv) . Then, we have the following identity*

$$\langle Z_{n;t}, G \rangle - \langle Z_{n;t}, G' \rangle = \sum_{j=1}^n \sum_{J \in \mathcal{J}_j} a_{J;j} \langle Z_{J;j}, G \rangle, \quad (6.8)$$

for some integer coefficients $a_{J;j}$ and where the index sets \mathcal{J}_j are given as follows

$$\begin{aligned} \mathcal{J}_t &= \{[1, i] \cup [k, n] \mid i = 1, \dots, n-1; i+1 < k \leq n+1; t \in [1, i] \text{ or } t \in [k, n]\} \\ &\quad \cup \{[1, k] \cup [i, n] \mid 0 \leq k < i-1; i \leq n; t \in [1, k] \text{ or } t \in [i, n]\}, \\ \mathcal{J}_j &= \{[k, \dots, m] \mid 1 \leq k \leq j; j \leq m \leq n; k < m\}, \quad j \neq t. \end{aligned} \quad (6.9)$$

where $[a, b] = \emptyset$ for $a > b$, and $[a, a] = \{a\}$.

Proof. For brevity, let $z(\cdot)$ denote $\langle Z_{n;t}, \cdot \rangle$. Our basic ingredient in the computation of $z(G) - z(G')$ in (6.8) is a computation of the difference

$$\langle A, G \rangle - \langle A, G' \rangle \quad \text{for any } A \in \mathcal{A}(n; t). \quad (6.10)$$

Since the diagrams G and G' differ only by the location of the arrow g ($g \sim (i, j)$) there are two basic cases; first, there exists an arrow $\alpha \in A$, $\alpha \sim (i, j)$, potentially matching g under an embedding of A in G . In the second case there is no arrow in A with coordinates (i, j) . In the first case, we may express (6.10), using (6.6), as

$$\langle A, G \rangle - \langle A, G' \rangle = (\langle A, G \rangle_{\alpha \rightarrow g} - \langle A, G' \rangle_{\alpha \rightarrow g'}) + (\langle A, G \rangle_{\alpha \not\rightarrow g} - \langle A, G' \rangle_{\alpha \not\rightarrow g'}). \quad (6.11)$$

Note that the second term of (6.11) vanishes, i.e.

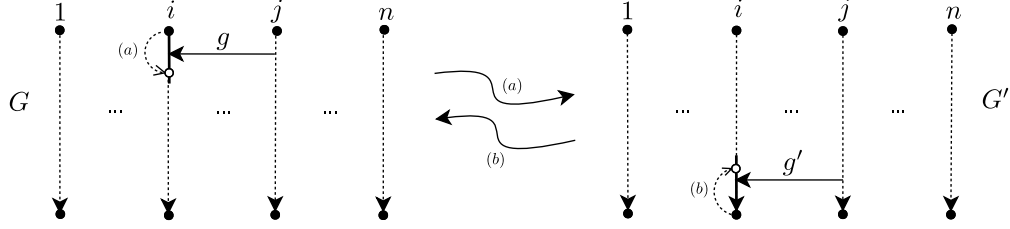
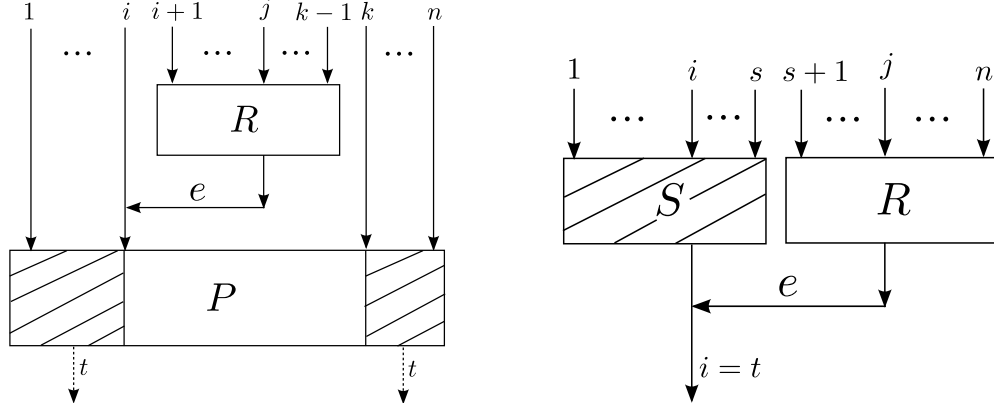
$$\langle A, G \rangle_{\alpha \not\rightarrow g} - \langle A, G' \rangle_{\alpha \not\rightarrow g'} = 0. \quad (6.12)$$

Indeed, for any embedding $\phi : A \hookrightarrow G$, such that $\phi(\alpha) \neq g$ (i.e. $\phi(A) \subset G - \{g\}$). Since $G - \{g\}$ and $G' - \{g'\}$ are identical we can define $\phi' : A \hookrightarrow G'$ as a composition of ϕ and the inclusion $G' - \{g'\} \subset G'$. Because $\text{sign}(\phi) = \text{sign}(\phi')$; we conclude $\langle A, G \rangle_{\alpha \not\rightarrow g} = \langle A, G' \rangle_{\alpha \not\rightarrow g'}$ which proves (6.12). If a diagram A has no arrow α matching g in G , the first term in the sum (6.11) vanishes, and $\langle A, G \rangle - \langle A, G' \rangle = \langle A, G \rangle_{\alpha \not\rightarrow g} - \langle A, G' \rangle_{\alpha \not\rightarrow g'}$. Then the same argument as for (6.12) implies $\langle A, G \rangle - \langle A, G' \rangle = 0$. Therefore, we obtain

$$\langle A, G \rangle - \langle A, G' \rangle = \langle A, G \rangle_{\alpha \rightarrow g} - \langle A, G' \rangle_{\alpha \rightarrow g'} \quad (6.13)$$

Next, we analyze in detail Cases (ι) – (ιv) illustrated on Figures 13–19.

Case (ι) : Suppose $g \in G$, $g \sim (i, j)$, $i < j$ has its arrowhead closest to the basepoint along the i th string as pictured on Figure 13, and G' is obtained from G by applying move (a) (or equivalently G is obtained from G' via the move (b)). The following cases, define certain disjoint subsets in $\mathcal{A}(n; t)$ of tree diagrams which may yield a nonzero right hand side of (6.13). Roughly speaking, Case $(\iota.b)$ concerns the diagrams which have an arrow $\alpha \sim (i, j)$ with $h(\alpha)$ at the top of component i , such diagrams possibly yield $\langle A, G \rangle_{\alpha \rightarrow g} \neq 0$ (Figure 14(left)). Case $(\iota.c)$ concerns diagrams having $\alpha \sim (i, j)$ with $h(\alpha)$ at the bottom of the i -component, such diagrams possibly contribute to $\langle A, G' \rangle_{\alpha \rightarrow g'}$ (Figure 14(right)). Case $(\iota.a)$ is a special subcase of $(\iota.c)$ and concerns diagrams with a single $\alpha \sim (i, j)$ on the i -component, this can only happen for $t = i = 1$ (by **(d1)**–**(d4)**).

FIGURE 13. Case (ι) : Basepoint moving pass the arrowhead of $g \sim (i, j)$, $i < j$.FIGURE 14. Diagrams in $(\iota.b)$ and $(i.c)$, where e corresponds to α , the “dashed” vertical arrows, in the left diagram, indicate a possible location for the trunk of A indexed by t .

Case $(\iota.a)$, (diagrams possibly contributing to both $\langle A, G \rangle_{\alpha \rightarrow g}$ and $\langle A, G' \rangle_{\alpha \rightarrow g'}$ in (6.13)): Define

$$\mathcal{A}_{(\iota.a)}(j) = \{A \in \mathcal{A}(n; 1) \mid \text{exists } \alpha \in A, \text{ such that } \alpha \sim (1, j), \text{ and } \alpha \text{ is the only arrow with its head on the first component of } A\} \quad (6.14)$$

(Figure 14(left) with $S = \emptyset$ shows the general form of such diagrams). Then given a representation $\phi : A \mapsto G$, $\phi(\alpha) = g$, we may define $\phi' : A \mapsto G'$ to differ from ϕ only by assigning g' to α , i.e. $\phi'(\alpha) = g'$. Because $\text{sign}(\phi) = \text{sign}(\phi')$, we obtain $\langle A, G \rangle_{\alpha \rightarrow g} = \langle A, G' \rangle_{\alpha \rightarrow g'}$, yielding zero on the left hand side of (6.13), and as a consequence;

$$\sum_{A \in \mathcal{A}_{(\iota.a)}(j)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) = 0. \quad (6.15)$$

Case $(\iota.b)$ (diagrams possibly contributing to $\langle A, G \rangle_{\alpha \rightarrow g}$ in (6.13)): Consider diagrams $A \in \mathcal{A}(n; k)$ with an arrow $\alpha \sim (i, j)$, which admit the following decomposition with respect to α (Figure 14(left) and Equation (5.13) with $S = \emptyset$, $P \neq \emptyset$)

$$A = P \prec_i Q, \quad Q = e \prec_2 R, \quad (6.16)$$

where α corresponds to e . Note, that there is no other arrowhead above the arrowhead of α along the i -component of A , hence $l(R) = j - i - 1$ (see notation in (5.3)) and because $P \neq \emptyset$, A cannot be in $\mathcal{A}_{(\iota.a)}(j)$.

For a fixed $I \subset [n]$ and $J = ([n] - I) \cup \{i\}$, denote a set of diagrams $A \in \mathcal{A}(n; t)$ admitting the decomposition (6.16) with $I = I(P; A)$ and $J = I(Q; A)$, as $\mathcal{A}_{(\iota.b)} = \mathcal{A}_{(\iota.b)}(i, j; I; t)$. Clearly, not every $I \subset [n]$ yields a nonempty $\mathcal{A}_{(\iota.b)}(i, j; I; t)$, we must have I and J as given by (5.4); with $k = i$

and $l(R) = j - i - 1$. For further convenience, we define

$$\mathcal{J}_{(\iota, b)}(i, j) = \{I \subset [n] \mid \mathcal{A}_{(\iota, b)}(i, j; I; t) \neq \emptyset\}.$$

Using (5.4) and (6.16) it follows that $I(P; A) = [1, i] \cup [k, n]$, for an appropriate $k > j$ (see Figure 14, the “shaded” part of P) and therefore

$$\mathcal{J}_{(\iota, b)}(i, j) = \{[1, i] \cup [k, n] \mid k > j\}. \quad (6.17)$$

For $A \in \mathcal{A}_{(\iota, b)}$, any embedding $\phi : A \mapsto G$, $\phi(\alpha) = g$ has two restrictions

$$\xi_I = \phi|_P : P \mapsto G(I), \quad \psi_J = \phi|_Q : Q \mapsto G(J), \quad \psi_J(\alpha) = g, \quad J = ([n] - I) \cup \{i\}.$$

Conversely, consider any embedding $\xi_I : P \mapsto G(I) \subset G$, and an arrow $\beta \in P$. If the image arrow: $\xi_I(\beta)$ has its head/tail on the i th string of G it must be below the head of g and since $I \cap J = \{i\}$, any pair of embeddings $\xi_I : P \mapsto G(I)$, $\psi_J : Q \mapsto G(J)$, $\psi_J(\alpha) = g$, yields the “joint” embedding

$$\phi = \xi_I \sqcup \psi_J : A \mapsto G, \quad A = P \prec_i Q.$$

Since $\text{sign}(\phi) = \text{sign}(\xi_I)\text{sign}(\psi_J)$, we have

$$\begin{aligned} \langle A, G \rangle_{\alpha \mapsto g} &= \sum_{\substack{\phi: A \mapsto G, \\ \phi(\alpha) = g}} \text{sign}(\phi) = \sum_{\substack{\xi_I \sqcup \psi_J: A \mapsto G, \\ \psi_J(\alpha) = g}} \text{sign}(\xi_I)\text{sign}(\psi_J) \\ &= \left(\sum_{\xi_I: P \mapsto G(I)} \text{sign}(\xi_I) \right) \left(\sum_{\substack{\psi_J: Q \mapsto G(J), \\ \psi_J(\alpha) = g}} \text{sign}(\psi_J) \right) = \langle P, G(I) \rangle \langle Q, G(J) \rangle_{\alpha \mapsto g} \end{aligned} \quad (6.18)$$

(using the notation in (6.7)). On the other hand, since α has a top arrowhead along the i -component of A , there must be at least one other arrowhead on the i -component. Axiom (d4) implies that there can be no embedding $\phi : A \mapsto G'$, such that $\phi(\alpha) = g'$ because there is no arrowhead/tail below the head of g' along the i th string of G' (Figure 13(right)). In turn we obtain $\langle A, G' \rangle_{\alpha \mapsto g'} = 0$, and from (6.13), (6.18), we may compute, for a fixed $I \in \mathcal{J}_{(\iota, b)}(i, j)$, $J = ([n] - I) \cup \{i\}$:

$$\begin{aligned} \sum_{A \in \mathcal{A}_{(\iota, b)}(i, j; I; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) &= \sum_{A \in \mathcal{A}_{(\iota, b)}(i, j; I; t)} \text{sign}(A) \langle A, G \rangle_{\alpha \mapsto g} \\ &= \sum_{\substack{P \in \mathcal{A}(p; r); \\ p = |I|}} \sum_{\substack{Q \in \mathcal{A}(q; 1) \\ Q = e \prec_2 R; q = |J|}} \text{sign}(P)\text{sign}(Q) \langle P, G(I) \rangle \langle Q, G(J) \rangle_{\alpha \mapsto g}. \end{aligned} \quad (6.19)$$

where we used the sign identity (5.7), and r is determined by (5.2) with $m = q$ and t given in (6.8). For any $I \in \mathcal{J}_{(\iota, b)}(i, j)$, Equation (1.4) implies

$$\sum_{\substack{P \in \mathcal{A}(p; r); \\ p = |I|}} \text{sign}(P) \langle P, G(I) \rangle = Z_{I; t}(G). \quad (6.20)$$

For this reason we will refer to P as a *free factor* in the decomposition (6.16). Setting $b_I = \sum_{\substack{Q \in \mathcal{A}(q; 1) \\ Q = e \prec_2 R; q = |J|}} \text{sign}(Q) \langle Q, G(J) \rangle_{\alpha \mapsto g}$, we obtain from (6.19) and (6.20):

$$\sum_{I \in \mathcal{J}_{(\iota, b)}(i, j)} \sum_{A \in \mathcal{A}_{(\iota, b)}(i, j; I; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) = \sum_{I \in \mathcal{J}_{(\iota, b)}(i, j)} b_I Z_{I; t}(G). \quad (6.21)$$

Case (ι, c) (diagrams possibly contributing to $\langle A, G' \rangle_{\alpha \mapsto g'}$ in (6.13)): Consider diagrams $A \in \mathcal{A}(n; t)$ which have an arrow α with $\alpha \sim (i, j)$, $i = t$ and the arrowhead at the bottom of the i -component

of A . By Lemma 5.3, such diagrams admit the following decomposition with respect to α (Figure 14(right) and Equation (5.13) with $P = \emptyset$)

$$A = U \prec_1 S, \quad U = e \prec_2 R, \quad (6.22)$$

where α is an arrow in e . Note that α is a bottom arrow, i.e. there is no other arrowhead below the head of α along the trunk of A . Because $\alpha \sim (t, j)$, we must have $l(R) + r(S) = j - t - 1$. As in the case $(\iota.b)$, for a given $I \subset [n]$ and $J = ([n] - I) \cup \{i\}$, we define the set of tree diagrams in $\mathcal{A}(n; t)$ admitting the above decomposition (6.22), with $I(U; A) = J$, $I(S; A) = I$ as $\mathcal{A}_{(\iota.c)}(t, j; I)$ and $\mathcal{J}_{(\iota.c)}(t, j) = \{I \subset [n] \mid \mathcal{A}_{(\iota.c)}(t, j; I) \neq \emptyset\}$. From (5.4) it follows that $I(S; A) = [1, k]$, for $t \leq k < j$, yielding

$$\mathcal{J}_{(\iota.c)}(t, j) = \{[1, k] \mid t \leq k < j\}. \quad (6.23)$$

Given $A \in \mathcal{A}_{(\iota.c)}(t, j; I)$, observe that, unless $S = \emptyset$ which is covered by Case $(i.a)$, there is no embedding $\phi : A \mapsto G$ with $\phi(\alpha) = g$, since there is no arrowhead above g in G along the t 'th string of G . As a result (6.13) simplifies to $\langle A, G \rangle - \langle A, G' \rangle = -\langle A, G' \rangle_{\alpha \mapsto g'}$. Using (6.22), every embedding $\phi' : A \mapsto G'$, $\phi'(\alpha) = g'$ restricts to subdiagrams U and S , yielding $\xi'_J : U \mapsto G'(J)$, $J = I(U; A)$ and $\psi'_I : S \mapsto G'(I)$, $\psi'_J(\alpha) = g'$, $I = I(S; A)$. Because, there is no arrowhead/tail below the head of g' in G' , every pair $\xi'_J : U \mapsto G'(J)$, $\psi'_I : S \mapsto G'(I)$, $\psi'_J(\alpha) = g'$, gives an embedding $\phi' = \xi'_J \sqcup \psi'_I : A \mapsto G'$, $\phi'(\alpha) = g'$. Analogous computation as in (6.18), shows that S is a free factor of decomposition (6.22) and

$$\langle A, G' \rangle_{\alpha \mapsto g'} = \langle S, G'(I) \rangle \langle U, G'(J) \rangle_{\alpha \mapsto g'}, \quad J = I(U; A), \quad I = I(S; A). \quad (6.24)$$

As in (6.19), given $I \in \mathcal{J}_{(\iota.c)}(t, j)$ we compute

$$\begin{aligned} \sum_{A \in \mathcal{A}_{(\iota.c)}(t, j; I)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) &= - \sum_{A \in \mathcal{A}_{(\iota.c)}(t, j; I)} \text{sign}(A) \langle A, G' \rangle_{\alpha \mapsto g'} \\ &= - \sum_{\substack{S \in \mathcal{A}(s; t); \\ s=|I|}} \sum_{\substack{U \in \mathcal{A}(u; 1) \\ U=e \prec_2 R; u=|J|}} \text{sign}(S) \text{sign}(U) \langle S, G(I) \rangle \langle U, G(J) \rangle_{\alpha \mapsto g'} = c_I Z_{I; t}(G), \end{aligned} \quad (6.25)$$

where $c_I = \sum_{\substack{U \in \mathcal{A}(u; 1) \\ U=e \prec_2 R; u=|J|}} \text{sign}(U) \langle U, G(J) \rangle_{\alpha \mapsto g'}$.

Coming back to the main argument in Case (ι) ; note that subsets $\mathcal{A}_{(\iota.a)}$, $\mathcal{A}_{(\iota.b)}$ and $\mathcal{A}_{(\iota.c)}$ of $\mathcal{A}(n; t)$ are disjoint. If A does not belong to their union, then either 1°: it has no arrow $\alpha \sim (i, j)$, or 2°: there exists $\alpha \in A$ such that $\alpha \sim (i, j)$, but α is neither the top nor bottom arrow of A . For such a diagram A , we easily obtain from (6.13): $\langle A, G \rangle - \langle A, G' \rangle = 0$. Consequently, in Case (ι) , based on (6.15), (6.21) and (6.25) we compute

$$\begin{aligned} z(G) - z(G') &= \sum_{A \in \mathcal{A}(n; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) \\ &= \sum_{i, j} \sum_{I \in \mathcal{J}_{(\iota.b)}(i, j)} \sum_{A \in \mathcal{A}_{(\iota.b)}(i, j; I; t)} \text{sign}(A) \langle A, G \rangle_{\alpha \mapsto g} \\ &\quad - \sum_j \sum_{I \in \mathcal{J}_{(\iota.c)}(t, j)} \sum_{A \in \mathcal{A}_{(\iota.c)}(t, j; I)} \text{sign}(A) \langle A, G' \rangle_{\alpha \mapsto g'} \\ &= \sum_{i, j} \sum_{I \in \mathcal{J}_{(\iota.b)}(i, j)} b_I Z_{I; t}(G) - \sum_j \sum_{I \in \mathcal{J}_{(\iota.c)}(t, j)} c_I Z_{I; t}(G). \end{aligned} \quad (6.26)$$

Proving Lemma 6.1 in Case (ι) , where the coefficients $a_{I; j}$ ought to be chosen as b_I or c_I above.

The proofs of remaining cases have analogous steps, in the following paragraphs we limit the amount of details giving the most relevant parts and emphasizing the differences.

Case (ι) : Suppose $g = (i, j)$ in G , $i > j$ has its arrowhead closest to the basepoint along the i -component (Figure 15), and G' is obtained from G by applying move (a) (or equivalently G is obtained from G' via the move (b)). The type of diagrams which may contribute to (6.13), are “mirror reflections” of diagrams described in Case (ι) .

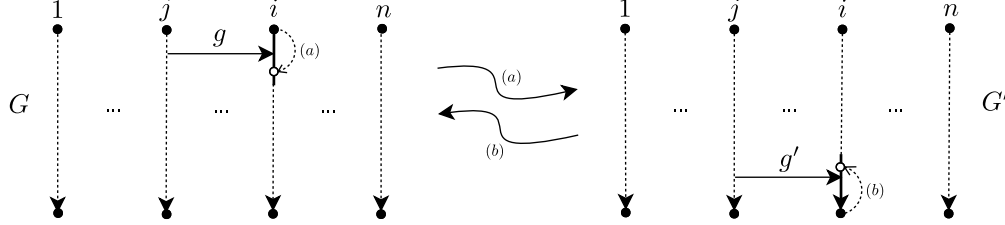


FIGURE 15. **Case (ι) :** Basepoint moving pass the head of $g \sim (i, j)$, $i > j$.

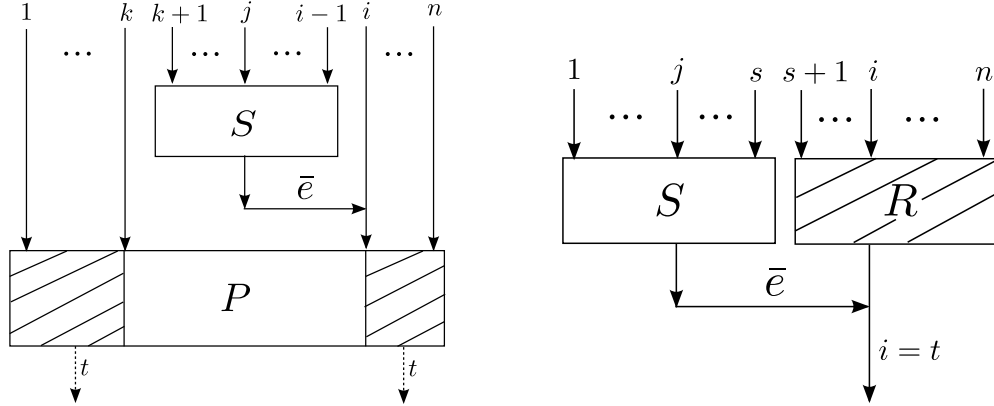


FIGURE 16. Diagrams in $(\iota.b)$ and $(\iota.c)$, where \bar{e} corresponds to α .

Case $(\iota.a)$: Diagrams $A \in \mathcal{A}(n; t)$ which may yield nonzero both $\langle A, G \rangle_{\alpha \rightarrow g}$ and $\langle A, G' \rangle_{\alpha \rightarrow g'}$ in (6.13) need to have an arrow $\alpha \sim (i, j)$ with head on i -component and no other arrow adjacent to that component. It implies that i -component is the trunk of A and $\langle A, G \rangle - \langle A, G' \rangle = \langle A, G \rangle_{\alpha \rightarrow g} - \langle A, G' \rangle_{\alpha \rightarrow g'} = 0$. In fact A admits the decomposition on Figure 16(right) with $R = \emptyset$.

Case $(\iota.b)$: Diagrams which may yield $\langle A, G \rangle_{\alpha \rightarrow g} \neq 0$ in (6.13) need to have an arrow $\alpha \sim (i, j)$ with a top arrowhead along the i -component and admit the following decomposition with respect to α (Figure 16(left) and Equation (5.14) with $R = \emptyset$):

$$A = P \prec_i Q, \quad Q = \bar{e} \prec_1 S, \quad P \neq \emptyset, \quad (6.27)$$

where α is the arrow in \bar{e} , $v = i(i, A; P)$ and $r(S) = i - j$. Given $I \subset [n]$, $J = ([n] - I) \cup \{i\}$, define $\mathcal{A}_{(\iota.b)}(i, j; I; t)$ to be the set of trees in $\mathcal{A}(n; t)$ decomposable according to (6.27) with $I(P; A) = I$, and $I(Q; A) = J$. As usual, we let $\mathcal{J}_{(\iota.b)}(i, j) = \{I \subset [n] \mid \mathcal{A}_{(\iota.b)}(i, j; I; t) \neq \emptyset\}$. From (5.4), it follows that $I(P; A) = [1, k] \cup [i, n]$, for $k < j$, giving

$$\mathcal{J}_{(\iota.b)}(i, j) = \{[1, k] \cup [i, n] \mid k < j\}. \quad (6.28)$$

Since P is the free factor in (6.27), for any $A \in \mathcal{A}_{(\iota.b)}(i, j; I; t)$:

$$\langle A, G \rangle_{\alpha \rightarrow g} = \langle P, G(I) \rangle \langle Q, G(J) \rangle_{\alpha \rightarrow g},$$

and since $P \neq \emptyset$; we also have $\langle A, G' \rangle_{\alpha \mapsto g'} = 0$ yielding

$$\sum_{A \in \mathcal{A}_{(\mathcal{U}.b)}(i, j; I; t)} \text{sign}(A) \langle A, G' \rangle_{\alpha \mapsto g} = d_I Z_{I; t}(G). \quad (6.29)$$

Case $(\mathcal{U}.c)$: Diagrams which may yield $\langle A, G' \rangle_{\alpha \mapsto g'} \neq 0$ in (6.13), must have no other head/tail below the head of α along the i -component of A . Such diagrams admit the following decomposition with respect to α (Figure 16(right) and Equation (5.13) with $P = \emptyset$, $R \neq \emptyset$)

$$A = U \prec_{s+1} R, \quad U = \bar{e} \prec_1 S, \quad S \in \mathcal{A}(s; r) \quad (6.30)$$

where α is an arrow corresponding to \bar{e} . Let $\mathcal{A}_{(\mathcal{U}.c)}(t, j; I)$ denote the set of trees in $\mathcal{A}(n; t)$ decomposable according to (6.30) with $I(R; A) = I$, and $I(U; A) = J$, and $\mathcal{J}_{(\mathcal{U}.c)}(t, j) = \{I \subset [n] \mid \mathcal{A}_{(\mathcal{U}.c)}(t, j; I) \neq \emptyset\}$. From (5.4) it follows that $I(R; A) = [s+1, n]$, for $1 < s+1 \leq t \leq n$ (Figure 16(right)) and

$$\mathcal{J}_{(\mathcal{U}.c)}(i, j) = \{[s+1, n] \mid 1 < s+1 \leq t \leq n\}. \quad (6.31)$$

Since R is the free factor in (6.30), for any $A \in \mathcal{A}_{(\mathcal{U}.c)}(t, j; I)$ we have $\langle A, G' \rangle_{\alpha \mapsto g'} = \langle R, G(I) \rangle \langle U, G(J) \rangle_{\alpha \mapsto g}$, and since $R \neq \emptyset$, we also have $\langle A, G' \rangle_{\alpha \mapsto g} = 0$. Using (6.13) yields

$$\sum_{A \in \mathcal{A}_{(\mathcal{U}.c)}(t, j; I)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) = -f_I Z_{I; t}(G). \quad (6.32)$$

Based on $(\mathcal{U}.a)$, (6.29) and (6.32), we obtain

$$\begin{aligned} z(G) - z(G') &= \sum_{A \in \mathcal{A}(n; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) \\ &= \sum_{i, j} \sum_{I \in \mathcal{J}_{(\mathcal{U}.b)}(i, j)} d_I Z_{I; t}(G) - \sum_j \sum_{I \in \mathcal{J}_{(\mathcal{U}.c)}(t, j)} f_I Z_{I; t}(G). \end{aligned} \quad (6.33)$$

Therefore, Lemma 6.1 is proven in Case (\mathcal{U}) , where the coefficients $a_{I; j}$ ought to be chosen as d_I or f_I above.

Case $(\mathcal{U}\mathcal{U})$: Suppose $g = (i, j)$ in G , $i < j$ has its arrowtail closest to the basepoint along the j th string as pictured on Figure 17, and G' is obtained from G by applying move (a) (or equivalently G is obtained from G' via the move (b)). As in the previous cases, we need to analyze the right hand side of (6.13).

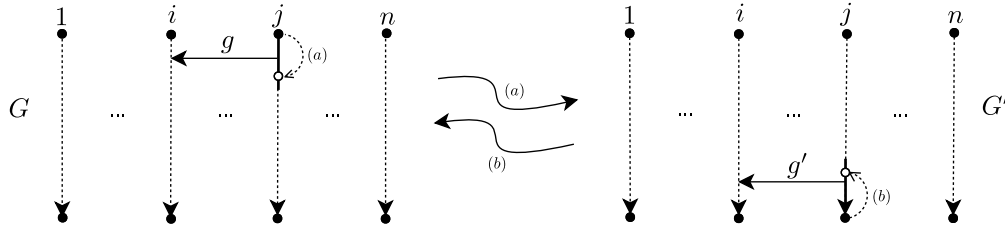
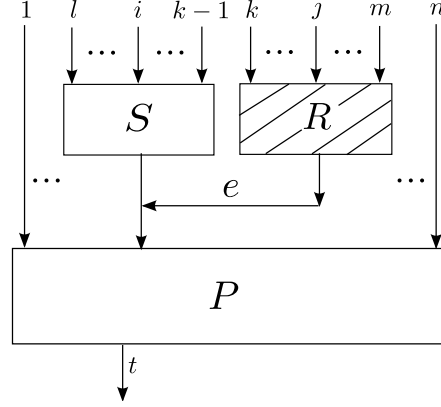


FIGURE 17. Case $(\mathcal{U}\mathcal{U})$: Basepoint moving pass the arrow tail of a top (i, j) arrow for $i < j$.

Case $(\mathcal{U}\mathcal{U}.a)$: Because there is no arrowhead/tail above the tail of g along the j -component of G , any tree $A \in \mathcal{A}(n; t)$ for which $\langle A, G \rangle_{\alpha \mapsto g} \neq 0$ admits the following decomposition (see Figure 18 with $R = \emptyset$)

$$A = P \prec_v Q, \quad Q = e \prec_1 S, \quad v = i(i, A; P).$$


 FIGURE 18. Diagrams in $(\mathcal{U}.a)$ and $(\mathcal{U}.b)$, where e corresponds to α .

Thus there is no other than α 's arrowhead/tail in A along the j -component of A , and terms in $\langle A, G \rangle_{\alpha \rightarrow g}$ match those in $\langle A, G' \rangle_{\alpha \rightarrow g'}$ yielding

$$\langle A, G \rangle_{\alpha \rightarrow g} - \langle A, G' \rangle_{\alpha \rightarrow g'} = 0. \quad (6.34)$$

Case $(\mathcal{U}.b)$: If $\langle A, G' \rangle_{\alpha \rightarrow g'} \neq 0$, A needs to have an arrow $\alpha \sim (i, j)$, with the arrow tail on the j -component. If α is the only arrow connected the j -component we obtain (6.34). Otherwise, A has to decompose as follows (Figure 18)

$$A = U \prec_w R, \quad U = P \prec_v (e \prec_1 S), \quad R \neq \emptyset, \quad v = i(i, U; P), \quad w = i(j, A; U), \quad (6.35)$$

where α corresponds to e and $r(S) + l(R) = j - i - 1$. Given $I \subset [n]$, such that $j \in I$, $J = [n] - I$, define $\mathcal{A}_{(\mathcal{U}.b)} = \mathcal{A}_{(\mathcal{U}.b)}(i, j; I; t)$ to be the set of trees in $\mathcal{A}(n; t)$ which can be decomposed according to (6.35) with $I(R; A) = I$, $I(U; A) = J$ and let $\mathcal{J}_{(\mathcal{U}.b)}(i, j) = \{I \subset [n] \mid \mathcal{A}_{(\mathcal{U}.b)}(i, j; I; t) \neq \emptyset\}$. From (5.4), it follows that $I(R; A) = [k, \dots, m]$, for $j \leq m \leq n$; $k > i$ yielding

$$\mathcal{J}_{(\mathcal{U}.b)}(i, j) = \{[k, \dots, m] \mid k \leq j \leq m \leq n; k < m; i < k\}. \quad (6.36)$$

The trunk of $A \in \mathcal{A}_{(\mathcal{U}.b)}$, has index: $t < l$, $t > m$ or $t = i$. Further, for any $A \in \mathcal{A}_{(\mathcal{U}.b)}$ we have $\langle A, G' \rangle_{\alpha \rightarrow g'} = \langle R, G(I) \rangle \langle U, G(J) \rangle_{\alpha \rightarrow g'}$, and since $R \neq \emptyset$, we have $\langle A, G \rangle_{\alpha \rightarrow g} = 0$. Therefore R is a free factor in (6.35) and

$$\sum_{A \in \mathcal{A}_{(\mathcal{U}.b)}(i, j; I; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) = -h_I Z_{I; j}(G). \quad (6.37)$$

Based on $(\mathcal{U}.a)$ and $(\mathcal{U}.b)$ we obtain

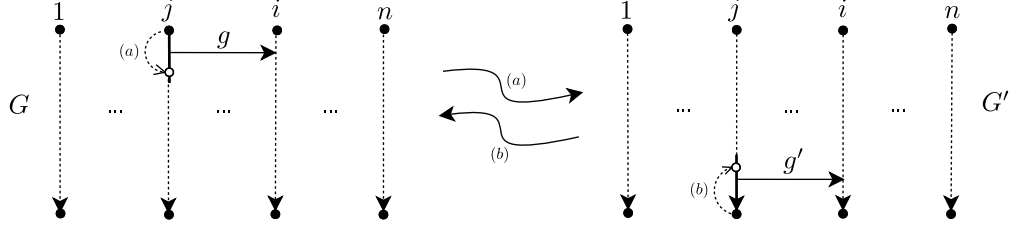
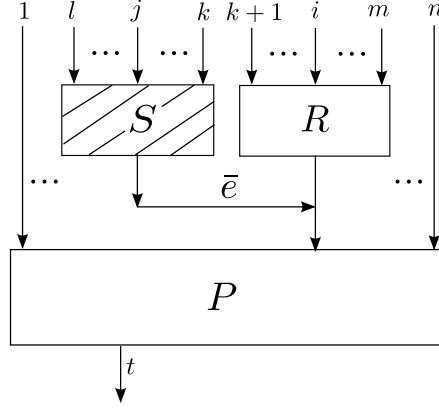
$$z(G) - z(G') = \sum_{A \in \mathcal{A}(n; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) = - \sum_{i, j} \sum_{I \in \mathcal{J}_{(\mathcal{U}.b)}(i, j)} h_I Z_{I; j}(G), \quad (6.38)$$

which ends the proof of Lemma 6.1 in Case (\mathcal{U}) .

Case (\mathcal{V}) : Suppose $g = (i, j)$ in G , $i > j$ has its arrowtail closest to the basepoint along the j th string as pictured on Figure 19, and G' is obtained from G applying move (a) (or equivalently G is obtained from G' via the move (b)).

Case $(\mathcal{V}.a)$: Any tree A giving $\langle A, G \rangle_{\alpha \rightarrow g} \neq 0$, admits the following decomposition, Figure 20 with $S = \emptyset$;

$$A = P \prec_v Q, \quad Q = \bar{e} \prec_2 R, \quad v = i(i, A; P).$$

FIGURE 19. Case (ν) : Basepoint moving pass the arrowtail of $g = (i, j)$ for $i > j$.FIGURE 20. Diagrams in $(\nu.a)$ and $(\nu.b)$, where \bar{e} corresponds to α .

As in Case $(\nu\nu.a)$, terms in the sum $\langle A, G \rangle_{\alpha \rightarrow g}$ match those in $\langle A, G' \rangle_{\alpha \rightarrow g'}$ yielding (6.34).

Case $(\nu.b)$: If $\langle A, G' \rangle_{\alpha \rightarrow g'} \neq 0$, A needs to have an arrow $\alpha \sim (i, j)$, with the bottom arrow tail on the j -component. Any relevant tree diagram A in this case not included in Case $(\nu.a)$ decomposes as pictured on Figure 20, i.e.

$$A = U \prec_w S, \quad U = P \prec_v (\bar{e} \prec_2 R), \quad S \neq \emptyset, \quad v = i(i - |S|, U; P), \quad w = i(j, A; U), \quad (6.39)$$

where α corresponds to \bar{e} and $r(S) + l(R) = j - i - 1$. Given $I \subset [n]$, such that $j \in I$, $J = [n] - I$, define $\mathcal{A}_{(\nu.b)} = \mathcal{A}_{(\nu.b)}(i, j; I; t)$ to be the set of trees which can be decomposed according to (6.39) with $I(S; A) = I$, $I(U; A) = J$ and $\mathcal{J}_{(\nu.b)}(i, j) = \{I \subset [n] \mid \mathcal{A}_{(\nu.b)}(i, j; I; t) \neq \emptyset\}$. From (5.4), it follows that $I(S; A) = [l, \dots, k]$, for $j \leq k < i$; $1 \leq l \leq j$, giving

$$\mathcal{J}_{(\nu.b)}(i, j) = \{[l, \dots, k] \mid 1 \leq l \leq j \leq k < i, l < k\}. \quad (6.40)$$

The trunk of A , has index $t < l$, $t > m$ or $t = i$. For any $A \in \mathcal{A}_{(\nu.b)}$ we obtain $\langle A, G' \rangle_{\alpha \rightarrow g'} = \langle S, G(I) \rangle \langle U, G(J) \rangle_{\alpha \rightarrow g'}$, and since $S \neq \emptyset$, we have $\langle A, G \rangle_{\alpha \rightarrow g} = 0$. Therefore, S is a free factor in (6.39) yielding

$$\sum_{A \in \mathcal{A}_{(\nu.b)}(i, j; I; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) = -w_I Z_{I; j}(G). \quad (6.41)$$

Based on $(\nu.a)$ and $(\nu.b)$, we obtain

$$z(G) - z(G') = \sum_{A \in \mathcal{A}(n; t)} \text{sign}(A) (\langle A, G \rangle - \langle A, G' \rangle) = - \sum_{i, j} \sum_{I \in \mathcal{J}_{(\nu.b)}(i, j)} w_I Z_{I; j}(G), \quad (6.42)$$

which ends the proof of Lemma 6.1 in Case (ν) .

Recapping Cases (ι) – $(\iota\nu)$ we obtain the identity (6.8) of Lemma 6.1, where the index sets \mathcal{J}_k are obtained from: (6.17), (6.23), (6.28), (6.36), (6.40). Specifically,

$$\mathcal{J}_t = \bigcup_{i,j} \mathcal{J}_{(\iota.b)}(i,j) \cup \mathcal{J}_{(\iota.c)}(t,j) \cup \mathcal{J}_{(\iota.b)}(i,j) \cup \mathcal{J}_{(\iota.c)}(t,j), \quad \mathcal{J}_j = \bigcup_i \mathcal{J}_{(\iota\iota.b)}(i,j) \cup \mathcal{J}_{(\iota\nu.b)}(i,j). \quad \square$$

6.2. Reidemeister moves. Figure 21 shows the Reidemeister moves: **r1** through **r3** of link diagrams away from the basepoints and the corresponding local arrow changes in their Gauss diagrams. A goal for this subsection is to prove the following

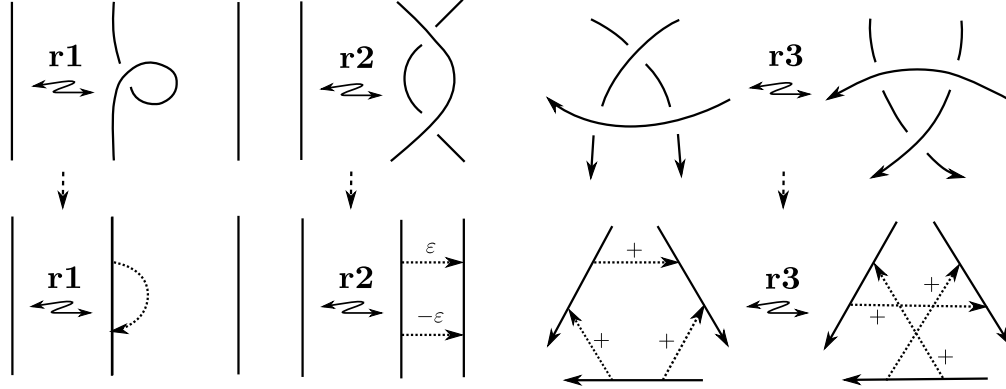


FIGURE 21. Reidemeister moves **r1**–**r3** locally on the link diagram (far from the basepoints) and the corresponding local changes of Gauss diagrams (c.f. [18]).

Lemma 6.2. *Given a Gauss diagram G of a closed based n -component link, let G' be a diagram obtained from G by applying locally one of the moves **r1**–**r3** away from the basepoints of link components. Then for any t we have*

$$\langle Z_{n;t}, G \rangle = \langle Z_{n;t}, G' \rangle.$$

The proof given below follows closely the argument of Kravchenko and Polyak [18] and is included mostly for completeness.

Proof of Lemma 6.2. Case r1: Let G be a Gauss diagram of a based link L , and G' a diagram obtained by the move **r1**, i.e. G and G' differ by a single arrow g shown on Figure 21. Since none of the tree diagrams $A \in \mathcal{A}(n; 1)$ have arrows with the heads and tails on the same segment, and there exists no embedding $\phi : A \hookrightarrow G$, such that $\phi(\alpha) = g$ for some $\alpha \in A$. Therefore every such embedding is also an into G' and we obtain $\langle A, G \rangle = \langle A, G' \rangle$, proving $z(G) = z(G')$ for **r1**.

Case r2: The diagram G' is obtained from G by adding locally two parallel arrows g_+ and g_- of opposite sign i.e. both arrows have their head on the i th string and the tail on the j th string of G' . By uniqueness of arrows in tree diagrams each embedding $\phi : A \hookrightarrow G'$ can map $\alpha \sim (i, j)$ in A to either g_+ or g_- , therefore

$$\langle A, G' \rangle = \langle A, G' \rangle_{\alpha \rightarrow g_+} + \langle A, G' \rangle_{\alpha \rightarrow g_-} + \langle A, G' \rangle_{\alpha \not\rightarrow \{g_+, g_-\}}. \quad (6.43)$$

Every $\phi' : A \hookrightarrow G'$, $\phi'(\alpha) \notin \{g_+, g_-\}$, factors through the inclusion $\iota : G' - \{g_+, g_-\} \hookrightarrow G'$, i.e. $\phi' = \iota \circ \phi$. Since $G = G' - \{g_+, g_-\}$: $\phi : A \hookrightarrow G$ is an embedding into G thus $\text{sign}(\phi') = \text{sign}(\phi)$ and we obtain $\langle A, G \rangle = \langle A, G' \rangle_{\alpha \not\rightarrow \{g_+, g_-\}}$. Next consider $\phi' : A \hookrightarrow G'$, $\phi'(\alpha) = g_+$, i.e. an embedding which may contribute to $\langle A, G' \rangle_{\alpha \rightarrow g_+}$, every such embedding can be redefined as $\phi : A \hookrightarrow G'$,

where $\phi|_{A-\{\alpha\}} = \phi'|_{A-\{\alpha\}}$ and $\phi(\alpha) = g_-$, since g_+ and g_- have opposite signs in G' we have $\text{sign}(\phi) = -\text{sign}(\phi')$. Therefore terms in $\langle A, G' \rangle_{\alpha \rightarrow g_+}$ are in one-to-one correspondence with terms in $\langle A, G' \rangle_{\alpha \rightarrow g_-}$, but with opposite sign, yielding $\langle A, G' \rangle_{\alpha \rightarrow g_+} + \langle A, G' \rangle_{\alpha \rightarrow g_-} = 0$. Collecting the above facts we obtain: $\langle A, G' \rangle = \langle A, G \rangle$, proving $z(G) = z(G')$ for **r2**.

Case r3: This is the most involved case, which includes subcases corresponding to the order of components i_1, i_2 and i_3 of G on which the **r3**-move is performed.

The rearrangement of arrows $g \sim (i_1, i_2)$, $h \sim (i_1, i_3)$ and $k \sim (i_2, i_3)$ under the **r3** move is shown on Figure 21. Given any tree diagram $A \in \mathcal{A}(n; t)$, let

$$\alpha \sim (i_1, i_2), \quad \beta \sim (i_1, i_3), \quad \gamma \sim (i_2, i_3). \quad (6.44)$$

Note that all three arrows cannot be in A , as it would contradict planarity of A . Let us determine those trees A , which can contribute to $\langle A, G \rangle - \langle A, G' \rangle$. Clearly, if A contains none of the arrows in (6.44) any embedding $\phi : A \mapsto G$ factors through the inclusion $G - \{g, h, k\} \hookrightarrow G$ and since $G - \{g, h, k\} = G' - \{g', h', k'\}$, ϕ also embeds A in G' then, analogously as in the case of **r2**, we conclude $\langle A, G \rangle = \langle A, G' \rangle$.

If A contains exactly one of the arrows in (6.44), without loss of generality, suppose $\beta \in A$, then we have an analog of (6.13):

$$\langle A, G \rangle - \langle A, G' \rangle = (\langle A, G \rangle_{\beta \rightarrow h} - \langle A, G' \rangle_{\beta \rightarrow h'}) + (\langle A, G \rangle_{\beta \not\rightarrow h} - \langle A, G' \rangle_{\beta \not\rightarrow h'}).$$

The second term of the above sum vanishes by the same reasoning as in the previous paragraph. The first term also vanishes, because every embedding $\phi : A \mapsto G$, $\phi(\beta) = h$ can be locally redefined as $\phi' : A \mapsto G'$, $\phi'|_{A-\{\beta\}} = \phi|_{A-\{\beta\}}$, $\phi'(\beta) = h'$ and $\text{sign}(\phi) = \text{sign}(\phi')$. The remaining

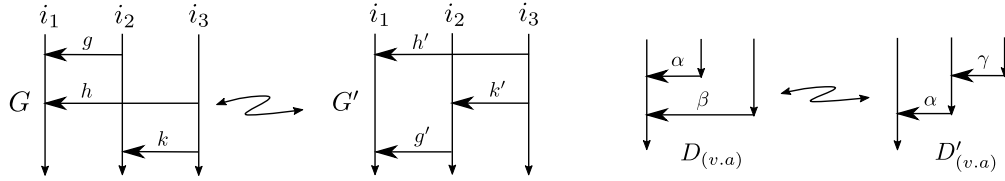


FIGURE 22. Case (v.a): Reidemeister **r3** for $i_1 < i_2 < i_3$. Matching arrows: $\{\alpha \rightarrow g, \beta \rightarrow h\}$, $\{\alpha \rightarrow g', \gamma \rightarrow k'\}$.

case is when A contains exactly two of the arrows from (6.44). The following cases depend on the ordering of strings in G .

Case (v.a): $i_1 < i_2 < i_3$ and either (a) : $\{\alpha, \beta\} \subset A$, (b) : $\{\alpha, \gamma\} \subset A$ or (c) $\{\beta, \gamma\} \subset A$.

First suppose A satisfies (a), given an embedding $\phi : A \mapsto G$, the only possibilities are 1°: $\phi(A) \subset G - \{g, h, k\}$, 2°: $\phi(\alpha) = g$ and $\phi(\beta) \neq h$ or $\phi(\alpha) \neq g$ and $\phi(\beta) = h$; 3°: $\phi(\alpha) = g$ and $\phi(\alpha) = h$. Therefore, we have

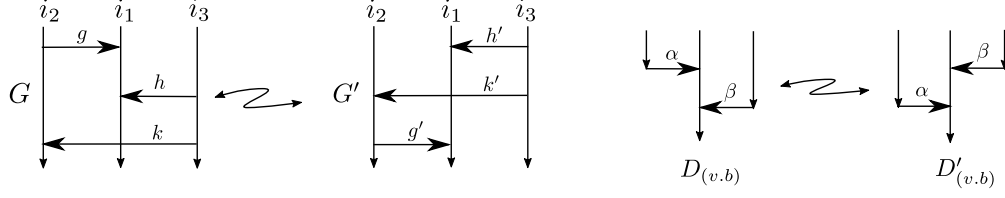
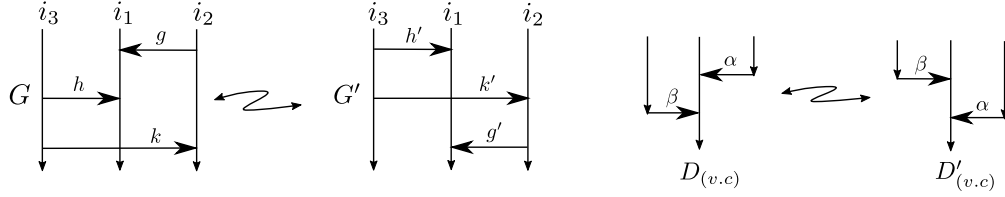
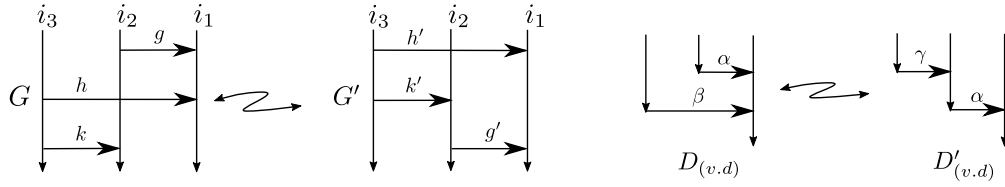
$$\langle A, G \rangle = \langle A, G \rangle_{\{\alpha, \beta\} \not\rightarrow \{g, h, k\}} + \langle A, G \rangle_{\alpha \rightarrow g, \beta \not\rightarrow h} + \langle A, G \rangle_{\alpha \not\rightarrow g, \beta \rightarrow h} + \langle A, G \rangle_{\alpha \rightarrow g, \beta \rightarrow h}. \quad (6.45)$$

On the other hand there is no embedding $\phi' : A \mapsto G'$, with $\phi'(\alpha) = g'$ and $\phi'(\beta) = h'$, and (6.45) becomes

$$\langle A, G' \rangle = \langle A, G' \rangle_{\{\alpha, \beta\} \not\rightarrow \{g', h', k'\}} + \langle A, G' \rangle_{\alpha \rightarrow g', \beta \not\rightarrow h'} + \langle A, G' \rangle_{\alpha \not\rightarrow g', \beta \rightarrow h'}. \quad (6.46)$$

It is clear that the first three terms of (6.45) and (6.46) agree, giving

$$\langle A, G \rangle - \langle A, G' \rangle = \langle A, G \rangle_{\alpha \rightarrow g, \beta \rightarrow h}. \quad (6.47)$$


 FIGURE 23. Case (v.b): Reidemeister **r3** for $i_2 < i_1 < i_3$.

 FIGURE 24. Case (v.c): Reidemeister **r3** for $i_3 < i_1 < i_2$.

 FIGURE 25. Case (v.d): Reidemeister **r3** for $i_3 < i_2 < i_1$.

In order for A to contribute to $\langle A, G \rangle_{\alpha \rightarrow g, \beta \rightarrow h}$, A must contain a subtree $D_{(v.a)}$ shown on Figure 22, (because the move is local, and there are no other arrows neighboring g, h, k). Denote the set of such diagrams by

$$\mathcal{A}_{D_{(v.a)}} = \{A \in \mathcal{A}(n; t) \mid D_{(v.a)} \subset A, I(D_{(v.a)}; A) = \{i_1, i_2, i_3\}\}.$$

In Case (b): $\{\alpha, \gamma\} \subset A$, fully analogous steps as in (6.45)–(6.47) yield

$$\langle A, G \rangle - \langle A, G' \rangle = -\langle A, G' \rangle_{\alpha \rightarrow g', \gamma \rightarrow k'}.$$

For A to contribute to $\langle A, G \rangle_{\alpha \rightarrow g', \gamma \rightarrow k'}$, it must contain a subtree $D'_{(v.a)}$ shown on Figure 22, denote the set of such diagrams by

$$\mathcal{A}_{D'_{(v.a)}} = \{A \in \mathcal{A}(n; t) \mid D'_{(v.a)} \subset A, I(D'_{(v.a)}; A) = \{i_1, i_2, i_3\}\}.$$

In Case (c) : $\{\beta, \gamma\} \subset A$, there is no embedding $\phi : A \rightarrow G$, with $\phi(\beta) = h$ and $\phi(\gamma) = k$, as well as no embedding $\phi' : A \rightarrow G'$, with $\phi'(\beta) = h'$ and $\phi'(\gamma) = k'$, yielding

$$\langle A, G \rangle - \langle A, G' \rangle = 0.$$

As a result we obtain

$$z(G) - z(G') = \sum_{A \in \mathcal{A}_{D_{(v.a)}}} \langle A, G \rangle_{\alpha \rightarrow g, \beta \rightarrow h} - \sum_{A \in \mathcal{A}_{D'_{(v.a)}}} \langle A, G' \rangle_{\alpha \rightarrow g', \beta \rightarrow k'}.$$

Observe that there is a bijection $f : \mathcal{A}_{D_{(v.a)}} \rightarrow \mathcal{A}_{D'_{(v.a)}}$, for a given $A \in \mathcal{A}_{D_{(v.a)}}$ defined simply by replacing $D_{(v.a)}$ subdiagram with $D'_{(v.a)}$. For every $A \in \mathcal{A}_{D_{(v.a)}}$, and $f(A) \in \mathcal{A}_{D'_{(v.a)}}$, an embedding $\phi : A \rightarrow G$ may be redefined as $\phi' : f(A) \rightarrow G'$, where $\phi'|_{f(A) - \{\alpha, \gamma\}} = \phi|_{A - \{\alpha, \beta\}}$ (since $f(A) -$

$\{\alpha, \gamma\} = A - \{\alpha, \beta\}$, and $\phi'(\alpha) = g'$ and $\phi'(\gamma) = k'$. Since $\text{sign}(D_{(v.a)}) = \text{sign}(D'_{(v.a)})$ we have $\text{sign}(\phi) = \text{sign}(\phi')$ (all arrows g, h, k in G , and g', h', k' in G' have the positive sign). In turn we obtain

$$\langle A, G \rangle_{\alpha \rightarrow g, \beta \rightarrow h} = \langle f(A), G' \rangle_{\alpha \rightarrow g', \beta \rightarrow k'}.$$

Since f is a bijection we may conclude that the right hand side of (6.50) vanishes and $z(G) = z(G')$ proving the claim in Case $(v.a)$.

Cases $(v.b)$: $i_2 < i_1 < i_3$, **$(v.c)$:** $i_3 < i_1 < i_2$, **$(v.d)$:** $i_3 < i_2 < i_1$. For these Cases the argument is the same as in Case $(v.a)$, except $\mathcal{A}_{D_{(v.*)}}$ and $\mathcal{A}_{D'_{(v.*)}}$ must be used in place of $\mathcal{A}_{D_{(v.a)}}$ and $\mathcal{A}_{D'_{(v.a)}}$ as shown on Figures 23, 24 and 25.

Cases $(vl.a)$: $i_1 < i_3 < i_2$, **and $(vl.b)$:** $i_2 < i_3 < i_1$; As before the two subcases (Figure 26 and Figure 27) are analogous, let us consider $(vl.a)$ in detail, see Figure 26. Either $(a) : \{\alpha, \beta\} \subset A$, $(b) : \{\alpha, \gamma\} \subset A$ or $(c) : \{\beta, \gamma\} \subset A$. In Case (a) , observe that for any $A \in \mathcal{A}(n; t)$, there is no embedding $\phi : A \hookrightarrow G$ such that $\phi(\alpha) = h, \phi(\beta) = g$. Using an analogous identity as Equation (6.45), we obtain

$$\langle A, G \rangle - \langle A, G' \rangle = -\langle A, G' \rangle_{\alpha \rightarrow g', \beta \rightarrow h'}. \quad (6.51)$$

The set of diagrams which may contribute to the left hand side in (6.51) is denoted by

$$\mathcal{A}_{D'_{(vl.a)}} = \{A \in \mathcal{A}(n; t) \mid D'_{(vl.a)} \subset A, I(D'_{(vl.a)}; A) = \{i_1, i_3, i_2\}\}.$$

In Case (b) there is no embedding which may contribute to $\langle A, G \rangle_{\alpha \rightarrow g, \gamma \rightarrow k}$ thus

$$\langle A, G \rangle - \langle A, G' \rangle = -\langle A, G' \rangle_{\alpha \rightarrow g', \gamma \rightarrow k'}. \quad (6.52)$$

$$\mathcal{A}_{D''_{(vl.a)}} = \{A \in \mathcal{A}(n; t) \mid D''_{(vl.a)} \subset A, I(D''_{(vl.a)}; A) = \{i_1, i_3, i_2\}\}.$$

In Case (c) there is no embedding of A which may contribute either to $\langle A, G \rangle_{\beta \rightarrow h, \gamma \rightarrow k}$ or to $\langle A, G' \rangle_{\beta \rightarrow h', \gamma \rightarrow k'}$. As a result we obtain

$$z(G) - z(G') = - \sum_{A \in \mathcal{A}_{D'_{(vl.a)}}} \langle A, G' \rangle_{\alpha \rightarrow g', \beta \rightarrow h'} - \sum_{A \in \mathcal{A}_{D''_{(vl.a)}}} \langle A, G' \rangle_{\alpha \rightarrow g', \gamma \rightarrow k'}. \quad (6.53)$$

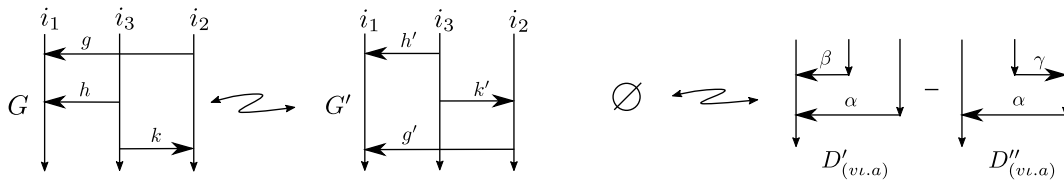


FIGURE 26. Case $(vl.a)$: Reidemeister **r3** for $i_1 < i_3 < i_2$.

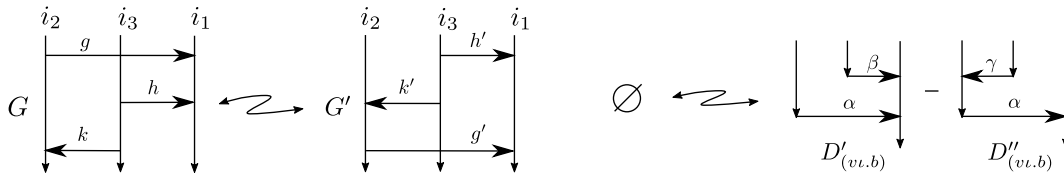


FIGURE 27. Case $(vl.b)$: Reidemeister **r3** for $i_2 < i_3 < i_1$.

Since diagrams in $\mathcal{A}_{D'_{(vl.a)}}(\mathcal{A}_{D''_{(vl.a)}})$ all must contain $D'_{(vl.a)}(D''_{(vl.a)})$ as a local subdiagram, we observe that there is a bijection $f : \mathcal{A}_{D'_{(vl.a)}} \hookrightarrow \mathcal{A}_{D''_{(vl.a)}}$ given by replacing $D'_{(vl.a)}$ in $A \in \mathcal{A}_{D'_{(vl.a)}}$

with $D'_{(v\iota.a)}$. Moreover, for any representation $\phi : A \mapsto G'$ there exists a corresponding $\phi' : f(A) \mapsto G'$, defined in the obvious way, and since $\text{sign}(D'_{(v\iota.a)}) = -\text{sign}(D''_{(v\iota.a)})$, we have $\text{sign}(\phi) = -\text{sign}(\phi')$, and therefore

$$\langle A, G' \rangle_{\alpha \rightarrow g', \beta \rightarrow h'} = -\langle f(A), G' \rangle_{\alpha \rightarrow g', \gamma \rightarrow k'}.$$

As a result the right hand side of (6.53) vanishes, and $z(G) = z(G')$ as required.

Cases above justify the claim of Lemma 6.2. \square

Proof of Main Theorem. For convenience, and without loss of generality, we will work with $I = [n]$ and $Z_{n;t}$. Let L be an n -component based link and $G = G_L$ its Gauss diagram, recall $Z_{n;t}(L) = \langle Z_{n;t}, G_L \rangle$, according to (1.4) and (1.3). Suppose $G' = G'_L$ is a diagram, obtained by moving basepoints along components of L in an arbitrary way. Naturally, there is a sequence of diagrams

$$G = G_0, G_1, \dots, G_k = G',$$

where G_i and G_{i+1} , $0 \leq i < k$ differ just by a single crossing pass as in Cases (ι) – (ιv) of Lemma 6.1. From Lemma 6.1 we obtain for each i :

$$\langle Z_{n;t}, G_i \rangle - \langle Z_{n;t}, G_{i+1} \rangle = \sum_{j=1}^n \sum_{J \in \mathcal{J}_j} a_{J;j} \langle Z_{J;j}, G_i \rangle.$$

Therefore, $\langle Z_{n;t}, G \rangle - \langle Z_{n;t}, G' \rangle = \sum_{i=0}^{k-1} (\langle Z_{n;t}, G_i \rangle - \langle Z_{n;t}, G_{i+1} \rangle)$ and substituting in the above equation, we may express $\langle Z_{n;t}, G \rangle - \langle Z_{n;t}, G' \rangle$ as a linear combination of $\{\langle Z_{I;j}, G_i \rangle \mid 0 \leq i < k\}$, i.e.

$$\langle Z_{n;t}, G \rangle - \langle Z_{n;t}, G' \rangle = \sum_{i=0}^{k-2} \sum_{J,j} a_{J;j,i} \langle Z_{J;j}, G_i \rangle, \quad a_{J;j,i} \in \mathbb{Z},$$

where the term $\langle Z_{I;j}, G_k \rangle$ is excluded from the sum. Applying this step and (6.8) inductively to terms in the sum above we conclude that $\langle Z_{n;t}, G \rangle - \langle Z_{n;t}, G' \rangle$ is a linear combination of integers from the set

$$\Gamma_Z = \{\langle Z_{J;k}, G \rangle \mid J \subsetneq [n], k \in J\}, \quad (6.54)$$

(note that the index sets defined in (6.9) of Lemma 6.1, subjected to the above inductive process yield all subsets $J \subsetneq [n]$ with trunks $k \in J$). Letting $\Delta_Z(n; t) := \gcd(\Gamma_Z)$, we conclude that

$$\langle Z_{n;t}, G_L \rangle \equiv \langle Z_{n;t}, G'_L \rangle \pmod{\Delta_Z(n; t)}, \quad (6.55)$$

and thus $\bar{Z}_{n;t}(L)$ is invariant under the basepoint changes.

Invariance of $\bar{Z}_{n;t}(G_L)$ under Reidemeister moves follows immediately from Lemma 6.2, if there are no basepoints on the interacting strands locally. If a move involves basepoints, we may slide them away obtaining a diagram G'_L , then using (6.55), Lemma 6.2 applied to G'_L yields the claim. The link homotopy invariance of $\bar{Z}_{n;t}(G_L)$ follows from the fact that tree diagram do not have arrows with a head and tail on the same component. Thus crossing changes within a given component of L , do not affect the value of $\bar{Z}_{n;t}(L)$. \square

Proof of Corollary 1.2. Recall that we claim

$$\bar{Z}_{n;1}(L) = \bar{\mu}_{n;1}(L). \quad (6.56)$$

After a possible link homotopy of L we may assume $L \simeq \widehat{\ell}$ for some string link ℓ , c.f. [11]. Using (3.8) and Theorem 4.3,

$$\bar{\mu}_{n;1}(L) = \langle Z_{n;1}, G_\ell \rangle \pmod{\Delta_\mu(n; 1)},$$

and from Main Theorem,

$$\bar{Z}_{n;1}(L) = \langle Z_{n;1}, G_\ell \rangle \mod \Delta_Z(n; 1).$$

Therefore, it suffices to prove

$$\Delta_Z(n; 1) = \Delta_\mu(n; 1). \quad (6.57)$$

For that purpose, first observe that by **(s1)** (Equation (4.4)) we may consider a subset $\Gamma'_Z \subset \Gamma_Z$:

$$\Gamma'_Z = \{ \langle Z_{J;k^+}, G \rangle, \langle Z_{J;k^-}, G \rangle \mid J \subsetneq [n], k^+ = \max(J), k^- = \min(J) \},$$

and $\Delta'_Z(n; 1) = \gcd(\Gamma'_Z)$, giving us

$$\Delta_Z(n; 1) = \Delta'_Z(n; 1).$$

Now, Corollary 4.11 and cyclic symmetry of $\bar{\mu}$ -invariants, yields (6.57), proving our claim. \square

Remark 6.3. We would like to emphasize that Main Theorem characterizes the value $\langle Z_{I;j}, G_L \rangle$ of the arrow polynomial $Z_{I;j}$ on an arbitrary Gauss diagram G_L of a closed based link L , and shows that the residue class is an invariant (this property is crucial in the forthcoming paper [15]). The invariant $\bar{Z}_{n;1}(L)$ by itself can be defined in an obvious way, via Equation (3.8) and Theorem 4.3 in terms of $Z_{n;1}(G_\ell)$ where $\hat{\ell}$ is link homotopic to L .

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